Arts Section

General Mathematics

second secondary grade

Student book second term

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Introduction

بسم الله الرحمن الرحيم

We are pleased to introduce this book to show the philosophy on which the academic content has been prepared. This philosophy aims at:

1. Developing and integrating the knowledgeable unit in Math, combining the concepts and relating all the school mathematical curricula to each other.
2. Providing learners with the data, concepts, and plans to solve problems.
3. Consolidate the national criteria and the educational levels in Egypt through:
 - **A)** Determining what the learner should learn and why.
 - **B)** Determining the learning outcomes accurately. Outcomes have seriously focused on the following: learning Math remains an endless objective that the learners do their best to learn it all their lifetime. Learners should like to learn Math. Learners are to be able to work individually or in teamwork. Learners should be active, patient, assiduous and innovative. Learners should finally be able to communicate mathematically.
4. Suggesting new methodologies for teaching through (teacher guide).
5. Suggesting various activities that suit the content to help the learner choose the most proper activities for him/her.
6. Considering Math and the human contributions internationally and nationally and identifying the contributions of the achievements of Arab, Muslim and foreign scientists.

In the light of what previously mentioned, the following details have been considered:

- ★ This book contains three domains: algebra, relations and functions, calculus and trigonometry. The book has been divided into related and integrated units. Each unit has an introduction illustrating the learning outcomes, the unit planning guide, and the related key terms. In addition, the unit is divided into lessons where each lesson shows the objective of learning it through the title You will learn. Each lesson starts with the main idea of the lesson content. It is taken into consideration to introduce the content gradually from easy to hard. The lesson includes some activities, which relate Math to other school subjects and the practical life. These activities suit the students' different abilities, consider the individual differences throughout Discover the error to correct some common mistakes of the students, confirm the principle of working together and integrate with the topic. Furthermore, this book contains some issues related to the surrounding environment and how to deal with.
- ★ Each lesson contains examples starting gradually from easy to hard and containing various levels of thoughts accompanied with some exercises titled Try to solve. Each lesson ends in Exercises that contain various problems related to the concepts and skills that the students learned through the lesson.
- ★ Each unit ends in Unit summary containing the concepts and the instructions mentioned and General exams containing various problems related to the concepts and skills, which the student learned through the unit.
- ★ Each unit ends in an Accumulative test to measure some necessary skills to be gained to fulfill the learning outcome of the unit.
- ★ The book ends in General exams including some concepts and skills, which the student learned throughout the term.

Last but not least. We wish we had done our best to accomplish this work for the benefits of our dear youngsters and our dearest Egypt.

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Unit One

Sequences and series

Unit introduction

Undoubtedly, mathematics helps to discover and represent the patterns which might be finite or infinite where they can be existed in daily life situations or can be formed to serve different purposes. A lot of patterns are digital and have different uses in daily life. Patterns are found in forms of sequences and series. In the modern studies, the patterns have been developed from the theoretical domain into the practical one in the fields of science, geometry and statistics. Computers are the first technological appliances that can be improved and interacted uniquely and peerlessly to be used in analyzing the most complicated math and physics problems in all the knowledgeable branches.

Unit outcomes

By the end of this unit and doing all the activities involved , the student should be able to:

- Identify the concept of sequences and distinguish between sequences and series.
- Identify the arithmetic sequences and deduce its general term in different forms.
- Find the arithmetic mean of an arithmetic sequence and insert a finite number of arithemetic means between two numbers.
- Find the sum of a finite number of the terms of an arithmetic sequence in different forms.
- Identify the geometric sequence and deduce its general term in different forms.
- Find the geometric mean of a geometric sequence
- Insert a finite number of geometric means between two numbers.
- Deduce the relation between the arithmetic mean and geometric mean of two different positive numbers.
- Find the sum of a finite number of the terms of a geometric sequence in different forms.
- Find the sum of an infinite number of terms of a geometric sequence.
- Function the arithmetic and geometric sequences to interpret some life problems such as overpopulation.
- Use computers to do operations required to solve life and math problems on sequence and series.

Key terms

- Function
- Term
- Finite sequence
- Infinite sequence
- Increasing Seqence
- Decreasing Seqence
- Series
- Summation notation (Σ)
- Arithmetic sequence
- Common differnce
- Arithmetic mean
- Arithmetic series
- Geometric sequence
- Common ratio
- Geometric mean
- Geometric series
- Infinite geometric series
- Infinity

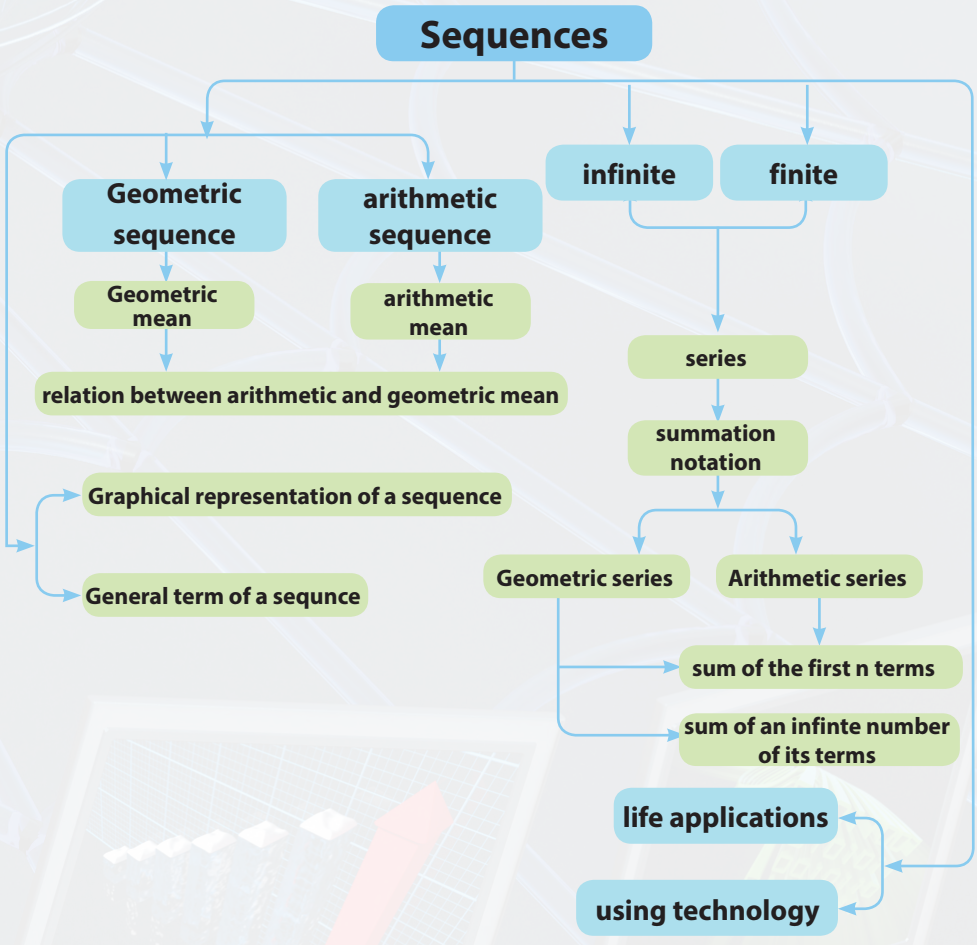
Unit Lessons

Lesson (1 - 1): Sequences and series.
Lesson (1 - 2): Arithmetic sequences.
Lesson (1 - 3): Arithmetic series.
Lesson (1 - 4): Geometric sequences.
Lesson (1 - 5): Geometric series.

Materials

- Scientifi calculator
- Graphics

Unit planning guide



Unit one

1 - 1 Sequences and Series

You will learn

- Definition of the sequence
- Finite and infinite sequence
- n^{th} term of the sequence
- Graphical representation of the sequence
- Series and summation notation

Key terms

- Sequence
- Finite Sequence
- Infinite Sequence
- Set
- Term
- Series
- Summation notation

Materials

- Scientifi calculator
- Graphics

Think and discuss

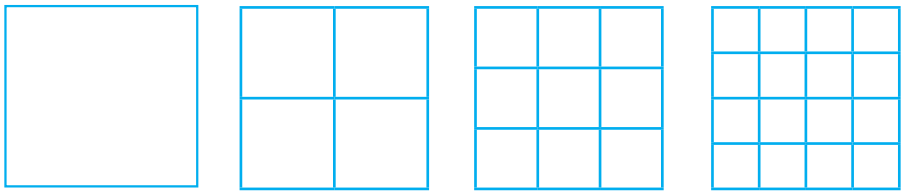


fig (1) fig (2) fig (3) fig (4)

The pattern above illustrates a square that can be divided into a number of smaller squares:

(1) find the number of small squares included in figure (5).
Draw figure (5).

(2) Can you find the number of small squares in figure 8?

(3) Can you find a relation between the number of small squares and the order of the figure?

Learn Sequence

The sequence is a function whose domain is a set of the positive integers Z^+ or a subset of it and its range is a set of the real numbers $\mathbb{R}$ where the first term is denoted by T_1, the second term is denoted by T_2 and the third term is denoted by T_3 and so on.... and the n^{th} term is denoted by Tn. The sequence can be expressed by writing down its terms between two brackets as follows: $(T_1, T_2, T_3, \ldots, T_n)$ or denoted by the symbol (T_n).

Remember

The function is a relation between the two sets X and Y so that each element of X appears as a first projection only one time in a limited ordered pairs of the relation.

Example

1 Write down the first six terms for each of the following sequences:

a The sequence of the positive even numbers which starts from the number (2)

b The sequence of the number included between 10 and 30 in which each of them is divisible by 3.

Notice

(1) Terms of a sequence are the images of the elements of the sequence domain.
(2) The symbol (T_n) expresses the sequence while the symbol T_n expresses its n^{th} term.

Solution

a (2 , 4 , 6 , 8 , 10, 12) b (12 , 15 , 18 , 21 , 24 , 27)

Try to solve

1 Write down the first six terms for each of the following equences:

a The sequcne of the negative odd numbers which starts from the number (- 1).

b The sequence of the numbers included between 51 and 81 in which each of them is divisible by 5.

General term of a sequence

The general term of a sequcne (it is sometimes called the n^{th} term) is written as T_n where T_n is the image of the element whose order is **n** in the domain of the sequence and it can be deduced through the given terms of the sequence.

For example:

- The general term of the sequence of the even numbers : 2 , 4 , 6 , 8 , ... is $T_n = 2n$
- The general term of the sequence of the odd numbers : 1 , 3 , 5 , 7 , ... is $T_n = 2n - 1$
- The general term of the sequence: $-\frac{1}{3}$, $\frac{1}{4}$, $-\frac{1}{5}$, $\frac{1}{6}$, ... is $T_n = \frac{(-1)^n}{n+2}$

Critical thinking: is there a rule to find the general term of all sequences ? Explain .

Example

2 Write down the first five terms of the sequence (T_n) defined as follows:
$T_1 = -1$ and $T_{n+1} = 2T_n$ where $n \geqslant 1$

Solution

By substituting the value of $n = 1 , 2 , 3 , 4$ in the relation $T_{n+1} = 2T_n$:

let $n = 1$, then $T_2 = 2T_1$ **i.e.:** $T_2 = 2 \times -1 = -2$ (substituting by $T_1 = -1$)

let $n = 2$, then $T_3 = 2T_2$ **i.e.:** $T_3 = 2 \times -2 = -4$ (substituting by $T_2 = -2$)

let $n = 3$, then $T_4 = 2T_3$ **i.e.:** $T_4 = 2 \times -4 = -8$ (substituting by $T_3 = -4$)

let $n = 4$, then $T_5 = 2T_4$ **i.e.:** $T_5 = 2 \times -8 = -16$ (substituting by $T_4 = -8$)

The first five terms of the sequence are: (- 1 , - 2 , - 4 , - 8 , - 16)

Try to solve

2 Write down the first six terms of the sequence (T_n) defined as follows:
$T_1 = 3$, $T_n = 2T_{n-1}$ where $n \geqslant 2$

Finite seqence and infinite sequence

The sequence is finite if the number of its terms is finite (**i.e. can be counted can not be counted**).
The sequence is infinite if the number of its terms is infinite (**an infinite number of elements can't be counted**).

Example

3 Write down each of the following sequences whose n^{th} term is given by the relation:

a $T_n = 2n - 1$ (to five terms starting from its first term).

b $T_n = n^2$ (to an infinite number of terms starting from its first term).

Solution

a Let n = 1 , 2, 3 , 4 , 5

$\therefore T_1 = 2(1) + 1 = 3$, $T_2 = 2(2) + 1 = 5$

$T_3 = 2(3) + 1 = 7$, $T_4 = 2(4) + 1 = 9$

$T_5 = 2(5) + 1 = 11$ $\therefore$ the sequence is: (3 , 5 , 7 , 9 , 11) **the sequence is finite**

b Let n = 1 , 2, 3 , 4 , 5 , ...

$\therefore T_1 = (1)^2 = 1$, $T_2 = (2)^2 = 4$

$T_3 = (3)^2 = 9$, $T_4 = (4)^2 = 16$

$T_5 = (5)^2 = 25$ $\therefore$ the sequence is: (1 , 4 , 9 , 16 , 25 , ...) **the sequence is infinite**

Try to solve

3 Write down each of the following sequences whose n^{th} term is given by the relation:

a $T_n = 1 - 3n$ (to five terms starting from its first term).

b $T_n = n^3$ (to an infinite number of terms starting from its first term).

Series and summation notation

The series is the adding operation of the sequence terms .

for example: (2 , 5 , 8 , 11 , ...) is a sequcne while 2 + 5 + 8 + 11 + ... is the series related to the previous sequence and the summation notation " Σ " can be used. It is read as (sigma) to write the series in a short form.

Example

4 Expand each of the following series , then find the sum of each expansion.

a $\sum_{r=1}^{4} r^2$ b $\sum_{r=3}^{7} (2r - 1)$

Solution

a let r = 1, then $T_1 = (1)^2 = 1$, let r = 2, then $T_2 = (2)^2 = 4$

let r = 3, then $T_3 = (3)^2 = 9$, let r = 4, then $T_4 = (4)^2 = 16$

i.e. the series is (1 + 4 + 9 + 16) and $\sum_{r=1}^{4} r^2 = 1 + 4 + 9 + 16 = 30$

b let r = 3, then $T_1 = 2 \times 3 - 1 = 5$, let r = 4, then $T_2 = 2 \times 4 - 1 = 7$

let r = 5, then $T_3 = 2 \times 5 - 1 = 9$, let r = 6, then $T_4 = 2 \times 6 - 1 = 11$

let r = 7, then $T_5 = 2 \times 7 - 1 = 13$

i.e. the series is (5 + 7 + 9 + 11 + 13) and $\sum_{r=3}^{7} (2r - 1) = 45$

Using the scientific calculator to find the sum of a series:



Scientific calculators help us do several complicated math problems quickly and accurately in a condition that the inputs must be correct . Finding the summation of a series is one of these essential operations . For example , we can check the summation of the series in the question (b) above as follows:

(1) Press the summation notation button $\sum$ (log) according to the specific color.

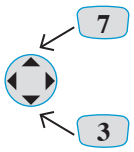
(2) Type the rule of the sequence (2r – 1) as follows:

start → 2 ALPHA) (x) – 1

(3) Use the button (Replay) to move as follows:

type the number of the terms of the sequence (7) in moving upwards,

type the order of the term by which we start with and its (3) in this example in moving downwards



(4) Press the button = to give the sum 45 and it is the same as the sum above.

Try to solve

4 Expand each of the following series, then find the sum of expansion and check the sum using the calculator.

a $\sum_{r=1}^{5} (3r - 2)$ **b** $\sum_{r=1}^{4} (r + 1)^2$ **c** $\sum_{r=5}^{9} 3 \times 2r^{-1}$

Exercises 1 - 1

Complete:

1. The fifth term of the sequence (T_n) where $T_n = 2n - 1$ is

2. The fourth term of the sequence (T_n) where $T_n = n^2 + 3$ is

3. In the sequence (T_n) where $T_{n+1} = n\,T_n$. If $T_1 = 1$, then T_2 = ...

Choose the correct answer:

4. The fifth term in the sequence of the natural numbers divisible by 5 is :
 a) 5 b) 25 c) 20 d) 10

5. The tenth term of the sequence whose n^{th} term is $T_n = \frac{2}{n} - 1$ where $n \in Z^+$ is:
 a) $\frac{-4}{5}$ b) $\frac{-1}{5}$ c) $\frac{1}{5}$ d) $\frac{4}{5}$

6. The rule of the sequence $((2 \times 3), (3 \times 4), (4 \times 5), (5 \times 6), ...)$ is:
 a) $(n - 1)(n + 1)$ b) $n(n + 1)$ c) $2n(n + 1)$ d) $(n + 1)(n + 2)$

Answer the following questions:

7. Show which of the following sequences is finite and which is infinite:
 a) $(1, 4, 7, 11, ...)$
 b) $(3, 5, 7, 9, ..., 21)$
 c) The sequence (T_n) where $T_n = n^2 - 1$, $n \in Z^+$
 d) The sequence (T_n) where $T_n = \frac{2}{n} + 3$, $n \in \{1, 2, 3, 4, 5\}$

8. Write down the first five terms for each of the sequences whose general term is given by the following rules:
 a) $T_n = n + n^2$ b) $T_n = \frac{1}{2n - 5}$ c) $T_n = (\frac{1}{3})^n$ d) $T_n = (-1)^n (n - 2)^2$

9. Discover the pattern , then write the next term
 a) $65, 69, 73, 77, 81, ...$ b) $3, -6, 12, -24, 48, ...$
 c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, ...$ d) $1, 3, 6, 10, 15, ...$

10. **Sports:** Kareem does physical fitness exercises for 8 minutes on the first day and increases the training period by the rate two minutes daily.
 a) Write the first five terms of this sequence.
 b) Find the general term of this sequence.
 c) How long does kareem take on the seventh day?
 d) On which day does kareem take half an hour? Explain.

11 Expand each of the following series:

a $\sum_{r=1}^{5} (3r - 2)$

b $\sum_{r=1}^{8} \left((-1)^r + 4r\right)$

c $\sum_{r=1}^{\infty} (\frac{1}{2})^{r-1}$

d $\sum_{r=1}^{\infty} (\frac{1}{r} - \frac{1}{r+1})$

12 Expand each of the following series, then find the sum of the expansion and check the result using the calculator.

a $\sum_{r=3}^{7} (2r + 3)$

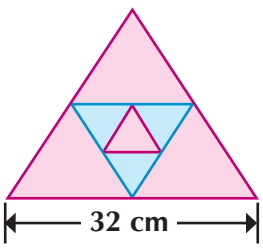
b $\sum_{r=2}^{6} (r^2 - 2)$

c $\sum_{r=1}^{5} 3 \times (\frac{1}{2})^{r+1}$

d $\sum_{r=3}^{6} (\frac{1}{r} + 2)$

13 **Mining:** A gold mine produced 4200 kg of gold in the first year and the production is decreased at a rate of 10 % yearly of the directly previous year. Write the first five terms of the production sequence , then find this sum.

14 **Geometry:** The figure opposite represents an equilateral triangle whose side length is 32cm. Its sides are bisected and the interior triangle is drawn. If this pattern lasts interiorly till we get three triangles including the first triangle.



a Write down the series representing the perimeters of the three triangles using the summation notation .

b Find the perimeter of the fourth triangle.

c Find the sum of the perimeters of the four triangles in this pattern in centimetres.

15 **Technology:** Kareem has electronically mailed a message for three friends, then each friend has mailed the same message to other three friends and so on .. Mailing the messages lasts the same pattern (known that each person has received the message once)

a Write down the series using the summation notation.

b How many persons do they receive the message in the fifth stage?

c Find the sum of all persons who will share this message till the fifth stage.

Unit one

1 - 2 Arithmetic Sequence

You will learn

- Definition of the arithmetic sequence.
- The graphical representation of the arithmetic sequence.
- n^{th} term of the arithmetic sequence.
- Identify the arithmetic sequence
- Insert a finite number of arithmetic means between two numbers.

Key terms

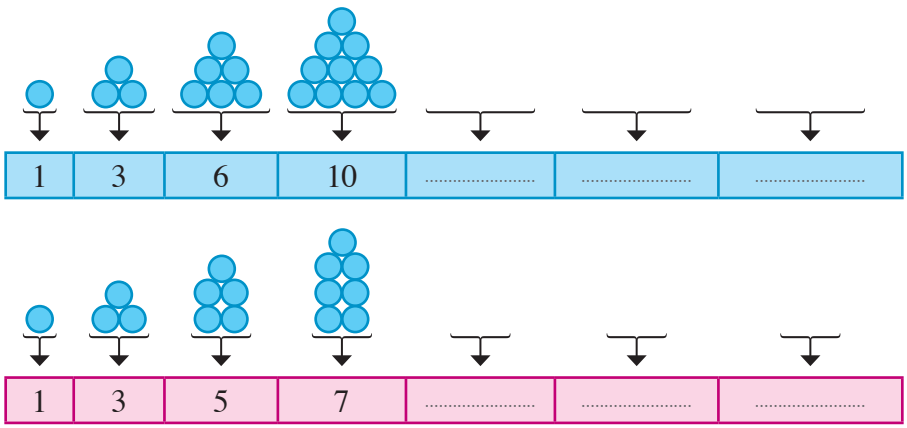
- Pattern
- Arithmetic sequence
- n^{th} Term
- Common difference
- Order of the term
- arithmetic mean

Materials

- Scientific Calculator
- Graphic programs

Activity

Study each of the following two patterns, then complete both of them till the seventh figure.



Answer the following questions:

(1) What are the aspects of similarity and difference between the first and second patterns?

(2) Write down the two sequences representing the two patterns above.

(3) What do you notice about the values of the sequence in the second pattern? Can you deduce a rule to relate the terms of this sequence? Write down this rule.

From the previous activity , we find that:

- The increase in the values of the elements of the first pattern varies, but the increase in the values of the elements of the second pattern is constant.
- The sequence representing the elements of the second pattern is: (1 , 3 , 5 , 7 , ...) where each element in it is greater than the directly previous term with a constant of 2. As a result, this sequence is called an arithmetic sequence.

Definition 1

Arithmetic Sequence

The arithmetic sequence is the sequence in which the difference between a term and the directly previous term equals a constant amount and it is called the common difference of the sequence. it is denoted by the symbol (d)

i.e.: $d = T_{n+1} - T_n$ for each $n \in Z^+$ and it can be formed in terms of its first term (a) and common difference (d).

Example

1 Which of the following sequences is an arithmetic sequence? Why?

a $(7, 10, 13, 16, 19)$ b $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6})$

c $T_n = 2n + 3$ d $T_n = \frac{3}{n} + 2$

Solution

a $\because T_2 - T_1 = 10 - 7 = 3$, $T_3 - T_2 = 13 - 10 = 3$

similarly $T_4 - T_3 = T_5 - T_4 = 3$

$\therefore T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = T_5 - T_4 = 3$ $\therefore$ the sequence is arithmetic and its common difference = 3

b $\because T_2 - T_1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$, $T_3 - T_2 = \frac{1}{4} - \frac{1}{3} = \frac{1}{12}$

$\therefore T_2 - T_1 \neq T_3 - T_2$ $\therefore$ the sequence is not arithmetic .

c $\because T_{n+1} - T_n = (2(n+1)+3) - (2n+3) = 2n + 2 + 3 - 2n - 3 = 2$

$\therefore$ the sequence is arithmetic and its common difference is 2

d $\because T_{n+1} - T_n = \left(\frac{3}{n+1} + 2\right) - \left(\frac{3}{n} + 2\right)$

$= \frac{3}{n+1} - \frac{3}{n} = \frac{3n - 3n - 3}{n(n+1)} = \frac{-3}{n(n+1)}$ doesnot equal a constant value

$\therefore$ the sequence is not arithmetic.

Try to solve

1 Which of the following sequences is an arithmetic sequence? Why?

a $(38, 33, 28, 23, 18)$ b $(-14, -8, -2, 4, 10)$

c $T_n = 2n + 3$ d $T_n = 1 - \frac{3}{2}n$

Increasing sequence and decreasing sequence

The arithmetic sequence (T_n) is increasing if its common difference is positive ($d >$ zero) such as: $(1, 5, 9, 13, \ldots\ldots)$

the arithmetic sequence (T_n) is decreasing if its common difference is negative ($d <$ zero) such as: $(4, -1, -6, -11, \ldots\ldots)$

Graphical representation of the arithmetic Sequence

2 Find the next four terms of the arithmetic sequence $(10, 7, 4, \ldots)$, then represent the first seven terms graphically.

Solution

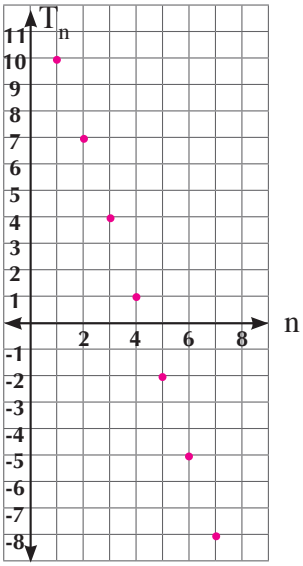
$\because d = T_2 - T_1 = 7 - 10 = -3$

$\therefore$ the next four terms are: 1 , - 2 , - 5 , - 8

the domain of the sequence is {1 , 2 , 3 , 4 , 5 , 6 , 7}

The range of the sequence is {10 , 7 , 4 , 1 , - 2 , - 5 , - 8}

The opposite figure shows the graphical representation of the sequence.



From the figure opposite , we notice that:

The points that represent the terms of the arithmetic sequence are collinear. This means that the arithmetic sequence is a function of the first degree in n where $n \in Z^+$ and the coefficient of n is the common difference of the sequence.

Critical thinking:

How do you expect the arithmetic sequence (T_n) if $(d = 0)$? Explain.

Try to solve

(2) In the sequence (T_n) where $T_n = 3n - 5$

(a) Prove that (T_n) is an arithmetic sequence and find its common difference .

(b) Show that this sequence is increasing.

(c) Find the fifteenth term of the sequence. (d) What's the value of n if $T_n = 85$?

Finding the n^{th} term of an arithmetic sequence:

From definition (1), the n^{th} term of the arithmetic sequence (T_n) whose first term is **a** and common difference is **d** can be deduced as follows :

$T_1 = a$, $T_2 = a + d$, $T_3 = a + 2d$

and by keeping this pattern, we find that the n^{th} term of this sequence is: $T_n = a + (n - 1)\, d$

Example

(3) In the arithmetic sequence (13 , 16 , 19 , ..., 100)

(a) Find the tenth term (b) Find the number of the terms of the sequence

Solution

$\because$ The sequence is arithmetic $\therefore a = 13$, $d = 16 - 13 = 3$

(a) $\because T_n = a + (n - 1)\, d$ $\therefore T_{10} = 13 + (10 - 1) \times 3$

$= 13 + 9 \times 3 = 13 + 27 = 40$

(b) The required is to find the value of n when $T_n = 100$

$\because T_n = a + (n - 1)\, d$ $\therefore 100 = 13 + (n - 1) \times 3$ $\therefore 100 = 13 + 3n - 3$

i.e.: $3n = 100 - 10 = 90$ $\therefore n = 30$

Try to solve

3 Find the number of the terms of the arithmetic sequence (7 , 9 , 11 , ..., 65), then find the value of the tenth term from the end.

Identfing the arithmetic Sequence:

The arithmetic sequence can be identified when its first term and common difference are known.

Example

4 Find the arithmetic sequence (T_n) in which $T_7 = 18$ and $T_{15} = 34$

Solution

From the given data , we know that : $T_7 = 18$, $T_{15} = 34$

$\because T_n = a + (n - 1)\, d$ $\quad \therefore 18 = a + (7 - 1)\, d$, then:

$\therefore a + 6d = 18$ **(1)**

similarly $34 = C + (15 - 1)\, d$

, $a + 14d = 34$ **(2)**

by solving the two equations **(1)**, **(2)** $\quad d = 2$

and by substituting in the first equation

$\therefore a + 6 \times 2 = 18$ $\quad \therefore a = 18 - 12 = 6$

$\therefore$ The arithmetic sequence is (6 , 8 , 10 , ...)

Notice

To get the value of d, then

a + 14d = 34

- a - 6d = - 18

by multiplying the two sides of the first equation by (- 1)

by adding the two equations: 8 d = 16

by dividing the two sides of the equation by 2

d = 2

Using the calculator:

To check the solution of the two equations: $a + 6d = 18$ and $a + 14d = 34$ use the calculator and follow the next procedures:



Entering data

Press the operation button MODE and choose EQN from the list by typing the number typed in front of it or by pressing the button EXE in some calculators, then choose the linear equation anX + bnY = cn by pressing the button.

We enter the factors of (X) , (Y) , and the absolute term (cn) respectively for the first equation then the second equation directly as follows:

start ⟶ 1 = 6 = 18 = 1 = 1 4 = 3 4 =

Recalling the sums:

- Press the button = first time to get the value of the first variable, let it be (X) and the result is X = 6
- Press the button = another time to get the value of the second variable, let it be (Y) and the result is Y = 2

To exist the program : Press the buttons: start ⟶ MODE 1

Try to solve

(4) Find the arithmetic sequence (T_n) in which $T_6 = 17$ and $T_3 + T_{10} = 37$

Arithmetic means:

You know that the arithmetic mean of the two numbers a and b is $\frac{a+b}{2}$

Considering that: (9 , 13 , 17 , 21 , 25) is an arithmetic sequence

- The arithmetic mean of the first and third terms = $\frac{9+17}{2} = 13$ **what do you notice?**
- The arithmetic mean of the second and fourth terms = $\frac{13+21}{2} = 17$ **what do you notice?**

Definition 2

If a , b and c are three consecutive terms in an arithmetic sequence, then b is called the arithmetic mean between the two numbers a and c where b- a = c-b,

i.e.: $2b = a + c$ then $b = \frac{a+c}{2}$ **So:** $\left(a , \frac{a+c}{2} , c\right)$ **is an arithmetic sequence.**

several arithmetic means: $x_1 , x_2 , x_3 , \dots , x_n$ can be inserted between the two numbers a and b in a way that : $(a , x_1 , x_2 , x_3 , \dots , x_n , b)$ **is an arithmetic sequence.**

Remember

The arithmetic mean of several quantities equals the sum of these numbers divided ny their numbers.

Example:

find the arithmetic mean of the numbers: 4 , 6 , 7 , 8 , 8 , 9

the arithmetic mean $= \frac{4+6+7+8+8+9}{6}$ **= 7**

Verbal expression: Complate:

(1) If you form (3, 7, 11, ..., 43 , 47) to be an arithmetic sequence , then : 7, 11, 15, ..., 43 is called

(2) The number of arithmetic means = the number of the terms of the sequence

(3) The number of the terms of the arithmetic sequence = the number of means of this sequence

Insert an infinite number of arithmetic means between two number

Example

(5) Insert 5 arithmetic means between 6 and 48

Solution

First: Find the number of the sequence terms

Find five terms between the first and the last terms in the sequence. Thus, the number of the terms of the arithmetic sequence is

$n = 2 + 5 = 7$

Second: Find the value of d:

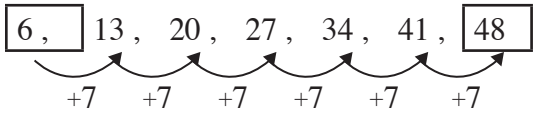
The n^{th} term of the arithmetic sequence is: $T_n = a + (n - 1) d$

by substituting: $a = 6 , T_n = 48 , n = 7$

$48 = 6 + (7 - 1) d$

i.e.: $6d = 42$ **by dividing the two sides by** 6 $\therefore d = 7$

Third: We use the value of d to find the required arithmetic means



The required means are: 13 , 20 , 27 , 34 , 41

Try to solve

5 Insert four arithmetic means between the two numbers 13 , 48

Exercises 1 - 2

Determine which of the following sequences are arithmetic and which are non-arithmetic, then find the common difference in case the sequence is arithmetic:

1 (12 , 15 , 18 , 21 , 24)

2 (21 , 25 , 29 , 34 , 38)

3 (-5 , -11 , -17 , -23 , -29)

4 (7 , 7 , 7 , 7 , 7)

5 (x + 2y, 3x + 3y, 5x + 4y) where x and y are two positive quantities

Write down the first five terms of the arithmetic sequence in each of the following cases:

6 a = 2 , d = 5

7 a = 7 , d = -3

8 a = -4 , d = $\frac{1}{4}$

Complete:

9 The seventh term of the arithmetic sequence (2 , 5 , 8 , ...) is

10 The eleventh term of the sequence (T_n) where T_n = 3n - 5 is

11 The n^{th} term of the arithmetic sequence (81 , 77 , 73 , ...) is

12 The n^{th} term of the arithmetic sequence ($\frac{1}{2}$, $\frac{1}{4}$, 0 , ...) is

13 The arithmetic mean of the two numbers 8 and 12 is

14 If the arithmetic mean of the two numbers *x* and 26 is 21, then *x* equals

Choose the correct answer from those given:

15 All the following sequences are arithmetic except:

a (3 , 7 , 11 , 15 , ...)

b (-11 , -15 , -19 , -23 , ...)

c ($\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, ...)

d ($\frac{21}{5}$, $\frac{16}{5}$, $\frac{11}{5}$, $\frac{6}{5}$, ...)

16 Of all the following sequences, the arithmetic sequence is:

a $(T_n) = (\frac{n+1}{n})$ b $(T_n) = ((n+1)^2)$

c $(T_n) = (\frac{3}{n}(n+2))$ d $(T_n) = \frac{n^3 - 1}{n^2 + n + 1}$

17 If (T_n) is an arithmetic sequence where $T_n = 3n + 2$, then the arithmetic mean between T_5 and T_{11} equals:

a 8 b 16 c 22 d 26

18 If $2a + 1$, $5a - 1$ and $6a + 3$ are three consecutive terms of an arithmetic sequence , then a equals.

a 1 b 2 c 3 d 4

Discover the error:

19 The common difference of the arithmetic sequence is defined as the difference between a term and the directly previous term i.e. $d = T_n - T_{n-1}$ for each $n \in Z^+$

20 It is not necessary that the points representing the terms of the arithmetic sequence are collinear.

21 The relation between n and T_n in the arithmetic sequence is given as follows: $T_n = an + b$ where a and b are two constants and b is the common difference of the sequence in this relation.

22 If we insert an arithmetic mean n between two consecutive numbers , then the number of the sequence terms equals n-2.

Answer the following questions:

23 Find the twelfth and twentieth terms of the arithmetic sequence (4 , 7 , 10 , ...)

24 Find the number of the terms of the arithmetic sequence (63 , 59 , 55 , ... , - 133)

25 Write down the first three terms of the sequence $(2 + 5n)$, then find the order of the term whose value is 72 in this sequence .

26 (T_n) is an arithmetic sequence in which $T_1 = -51$ and $T_{22} = -156$, find the common difference of this sequence.

27 Find the arithmetic sequence whose fourth term = 18 and the seventh term = 27.

28 What is the value of n in the arithmetic sequence whose first term = 3 , $T_n = 39$ and $T_{2n} = 79$? find the sequence.

29 Find the arithmetic sequence whose fifth term = 21 and its tenth term = 3 times its second term.

30 (T_n) is an arithmetic sequence in which $T_1 + T_2 = 9$ and $T_5 = 22$. Find this sequence

31 Find the arithmetic sequence whose sixth term = 20 and the ratio between the fourth and tenth terms is 4: 7.

32 Find the arithmetic sequence whose fourth term = 11 and the sum of the fifth and ninth terms = 40, then find the order of the term whose value is 152 in this sequence .

33 If 36 , a , 24 , b are consecutive terms of an arithmetic sequence, find the value of a and b.

34 If the arithmetic mean between a and b is 8 and the arithmetic mean between 4 a and 2 b is 20, find the value of a and b.

35 If (8 , a , ... , b , 68) form an arithmetic sequence of 16 terms, find the values of a and b

36 Insert 16 arithmetic means between 27 and - 24

37 In the arithmetic sequence (5 x + 6 , 3 x + 15 , ... , 50 x – 9 , 40 x + 16), find the value of x then find the number of the terms of this sequence.

38 **Physics:** Kareem has started to ride his motorcycle from the highest point of a downhill. He covered a distance of 100 cm in the first second . In each next second, he was covering a distance that increases in a magnitude of 120 cm from the directly previous distance. Find the distance, which he covers in the tenth second . .

39 **Trade:** A man has bought a motorcycle and he has agreed with the seller to pay its price in monthly installments forming an arithmetic sequence whose n^{th} term is 120 n + 80. If the last installment was 1400L.E., find the number of the installments .

40 **Creative thinking:** Find the arithmetic sequence in which the ninth term equals 25 and the arithmetic mean between its third and fifth terms is 10 .

Activity

Health: A patient takes 21 pills a week and his doctor recommended him to reduce taking these pills at a rate of 3 pills in the next week to the directly previous week.

1 How many pills will the patient take in the fifth week?

2 After how many weeks will the patient stop completely taking the pills?

3 How would you advise the patients who take medical drugs without getting a prescription of their doctors?

Unit one

1 - 3 Arithmetic Series

You will learn

- The concept of an arithmetic series
- Find the sum of n terms of an arithmetic sequence interms of its first and last terms .
- Find the sum of n terms of an arithmetic seuence in terms of its first term and the common difference.

Key terms

- Arithmetic series
- Summation notation (Σ)

Materials

- Scientific calculator

The sum of arithmetic series

The German scientist (Karl Gauss) had astonished his teacher when he was seven years old when he discovered the sum of adding the numbers from 1 to 100 mentally and in a fast way where he had noticed that the sum equals 50 pairs of the numbers which the sum of each is 101.

I.e.: $50 \times 101 = 5050$

Can you find the sum of the numbers from 1 to 20 mentally?



German scientist
Karl -Gauss
1777 - 1855

Definition

Arithmetic series

The arithmetic series is the operation of adding the terms of an arithmetic sequence.

For example: the sum of the first five terms of arithmetic sequence (3 , 5 , 7 , 9 , 11) is written as $S_5 = 3 + 5 + 7 + 9 + 11$

Sum of first n terms of an arithmetic series

First: finding the sum of first n terms of an arithmetic series in terms of its first and last terms.

If we have an arithmetic series of first term a, common difference d, last term ℓ and number of terms n, then the sum of n terms of this series is denoted by the symbol S_n where:

$s_n = a + (a + d) + (a + 2d) + ... + (\ell - d) + \ell$ **(1)**

This sum can be expressed in another form as follows:

$s_n = \ell + (\ell - d) + (\ell - 2d) + ... + (a + d) + a$ **(2)**

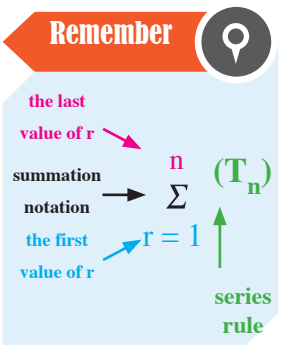
By adding the two equations (1) , (2) we deduce:

$2s_n = (a + \ell) + (a + \ell) + (a + \ell) + ... + (a + \ell)$ to n times

i.e. $2s_n = n(a + \ell)$ **by dividing the two sides by 2** $s_n = \frac{n}{2}(a + \ell)$

Example

1 **Using the summation notation Σ :** find $\sum_{r=5}^{24} (4r - 3)$



Solution

Because the expression inside the summation notation is of first degree, it represents an arithmetic sequence

$n = 24 - 5 + 1 = 20$ **Find the number of the sequence terms**

$T_n = 4n - 3$ **n^{th} term of the sequence**

$T_5 = 4 \times 5 - 3 = 17$, $T_{24} = 4 \times 24 - 3 = 93$

$s_n = \frac{n}{2}(a + \ell)$ **summation formula**

$s_{20} = \frac{20}{2}(17 + 93) = 1100$ **by substituting: a = 17 , ℓ = 93 , n = 20**

Try to solve

1 Find:

a $\sum_{k=1}^{20} (6k + 5)$ **b** $\sum_{m=7}^{32} (12 - 5m)$

Example

2 Find the sum of the arithmetic series $2 + 5 + 8 + ... + 62$

Solution

$\ell = a + (n - 1)d$ **n^{th} term of the sequence**

$62 = 2 + (n - 1) \times 3$ **by substituting a = 2, d = 3, ℓ = 62**

i.e.: $3n - 3 + 2 = 62$

$3n - 1 = 62$ **then n = 21**

$s_n = \frac{n}{2}(a + \ell)$ **summation formula**

$s_{21} = \frac{21}{2}(2 + 62) = 672$ **by substituting a = 2, n = 21, T_n = 62**

Try to solve

2 Find:

a The sum of the arithmetic series $89 + 85 + 81 , ... + 33$

b The number of the terms of the arithmetic sequence whose first term equals 3, last term equals 39 and the sum of the first n terms equals 210 .

Second: Finding the sum of n terms of an arithmetic series in terms of its first term and the common difference

You know that $\ell = a + (n - 1) d$, $s_n = \frac{n}{2}(a + \ell)$

by substituting from the first relation in the second relation, then:

$s_n = \frac{n}{2}[a + a + (n - 1) d]$ **i.e.:** $s_n = \frac{n}{2}[2a + (n - 1) d]$

Example

3 In the arithmetic series $5 + 8 + 11 + \ldots$ find :

a The sum of the first twenty terms of the series.

b The sum of ten terms starting from the seventh term.

c The sum of the series terms starting from T_{10} to T_{20}

Solution

$a = 5 \ , \quad d = 8 - 5 = 3$

a $s_n = \frac{n}{2} [2a + (n - 1) d]$ **summation formula**

$s_{20} = \frac{20}{2} \times [2 \times 5 + (20 - 1) \times 3]$ **by substituting a = 5 and d = 8 - 5 = 3**

$s_n = 10 \ (10 + 19 \times 3)$

$= 10 \times 67 = 670$ **by simplifying**

b $T_n = a + (n - 1) d$ **n^{th} term of the sequence**

$T_7 = a + 6d$

$= 5 + 6 \times 3 = 23$ **by substituting a = 5 , d = 3 and n = 7**

$s_{10} = \frac{10}{2} \times [2 T_7 + (10 - 1) \times 3]$ **by substituting in summation formula**

$s_{10} = 5 \times [2 \times 23 + 27]$

$= 5 \times 73 = 365$ **by simplifying**

c The sum of the sequence terms starting from T_{10} to T_{20}

$T_n = a + (n - 1) d$ **n^{th} term of the sequence**

$T_{10} = a + 9 d$

$= 5 + 9 \times 3 = 32$ **by substituting a = 5 , d = 3**

$T_{20} = a + 19 d = 5 + 19 \times 3 = 62$

$s_n = \frac{n}{2} (a + \ell)$ **summation formula (n = 20 - 10 + 1 = 11)**

$s_{11} = \frac{11}{2} (T_{10} + T_{20})$

$= \frac{11}{2} (32 + 62) = 517$ **by substituting $T_{10} = 32$, $T_{20} = 62$, n = 11**

Think:

Are there other solutions to find the sum of the sequence terms starting from T_{10} to T_{20}?

Try to solve

3 In the arithmetic series $9 + 12 + 15 + \ldots\ldots$ find:

a The sum of its first fifteen terms.

b The sum of the series terms starting from the fifth term up to the fifteenth term.

c The number of terms whose sum equals 750 starting from the first term.

Constructing the arithmetic sequence

Example

(4) Find the arithmetic sequence in which: $T_1 = 11$, $T_n = 87$ and $s_n = 980$

Solution

(a) Finding the value of n

$s_n = \frac{n}{2}(a + \ell)$ — **summation formula**

$980 = \frac{n}{2}(11 + 87)$ — **by substituting $T_1 = 11$, $T_n = 87$ and $S_n = 980$**

$98 \times \frac{n}{2} = 980$ then: $n = 20$ terms — **by simplifying $T_{20} = 87$**

(b) Finding the value of d

$T_n = a + (n - 1)\,d$ — **n^{th} term**

$87 = 11 + 19\,d$ — **by substituting $T_1 = 11$, $n = 20$ and $T_n = 87$**

$19d = 87 - 11 = 76$ — **by simplifying and by dividing by 19**

$\therefore d = 4$

(c) Constructing the sequence: $T_2 = 11 + 4 = 15$, $T_3 = 15 + 4 = 19$

The arithmetic sequence is (11 , 15 , 19 , ……, 87)

Try to solve

(4) Find the arithmetic sequence in which

(a) $T_1 = 23$, $T_n = 86$, $s_n = 545$

(b) $T_1 = 17$, $T_n = -95$, $s_n = -585$

Example

(5) **Investment:** In january first 2015, Kareem deposited 120 L.E in the postal saving office and on first of each month, he increases the amount he deposited for 15 L.E more than the directly previous month.

(a) Find the amount of money he deposited on December first in the same year.

(b) Find the sum of what he deposited starting from January first till end of December in the same year.

Solution

The amounts of money that Kareem deposited represent an arithmetic sequence whose first term is $a = 120$, the common difference $d = 15$ and the period of time is from January first till end of December 2015 where $n = 12$

(a) By substituting: $a = 120$, $d = 15$, $n = 12$ in the rule : $T_n = a + (n - 1)\,d$

$T_{12} = a + (12 - 1)d$ then $T_{12} = 120 + 11 \times 15 = 285$ L.E.

(b) By substituting: $a = 120$, $E = 15$, $n = 12$ in the rule $S_n = \frac{n}{2}[2a + (n - 1)\,d]$

$s_{12} = \frac{12}{2}(2 \times 120 + 11 \times 15) = 2430$ L.E .

Think: Can you find the total amount of money that Kareem deposited during the year using the rule of the summation of the arithmetic sequence in terms of its first and last terms and the number of terms? Explain to find the sum.

Try to solve

5 **Annual salary:** Kareem has started a job for an annual salary 19200 L.E. How much is kareem going to get at the end of the tenth year, if he gets an annual raise 480L.E?

Exercises 1 - 3

Complete:

1 The sum of the consecutive integers which starts from number 1 and ends in the number 20 equals

2 The sum of the first ten even numbers in the set of the natural numbers equals

3 The set of the odd natural numbers which is greater than 10 and less than 30 equals

4 The set of the natural numbers divisible by three and included between 30 and 50 equals

.....

5 The sum of the first nine terms of the arithmetic sequence whose first term is 2 and last term is 18 is.....

6 $\sum_{k=1}^{5} (2k + 1) =$.....

Choose the correct answer

7 The value of the arithmetic series $\sum_{r=1}^{4} (2r + 1)$ equals:

a 25 b 30 c 35 d 24

8 The formula of the series: $4 + 9 + 14 + ... + 5n - 1$ by using the summation notation equals:

a $\sum_{r=4}^{n} (5r - 1)$ b $\sum_{r=1}^{n} (5r - 1)$ c $\sum_{r=1}^{n} (5r + 1)$ d $\sum_{r=1}^{5n-1} (3r + 1)$

9 The formula of the series: $7 + 12 + 17 + 22$ by using the summation notation equals:

a $\sum_{r=1}^{4} (5r + 2)$ b $\sum_{r=1}^{4} (4r + 3)$ c $\sum_{r=1}^{4} (7r + 1)$ d $\sum_{r=1}^{4} (3r + 4)$

Discover the error

10 The sum of the terms of the arithmetic sequence (-12 , - 8 , - 4 , ... , 12) gives a positive integer.

11 The sum of the first six terms only of the arithmetic sequence (24 , 16 , 8 , ... , -24) equals zero.

Answer the following questions

12 Find the sum of first ten terms of the arithmetic sequence (14 , 18 , 22 , ...)

13 Find the sum of first fifteen terms of the arithmetic sequence whose first term is 4 and the fifteenth term is 26 .

14 Find the sum of the even numbers from 2 up to 40 .

15 Find the sum of the first 20 terms of the arithmetic series (6 + 4 + 2 + ...).

16 Find the sum of first thirty terms of the sequence (T_n) where $T_n = (2n + 3)$

17 Find the sun of the terms of the arithmetic sequence (2 , 5 , 8 , ... , 80).

18 Find the number of the terms necessary to be taken off from the sequence (16 , 20 , 24 , ...) starting from the first term to get a sum equal to 456.

19 How many terms are needed to be taken off from the sequence (- 16 , - 14 , -12 , ...) starting from its first term to get a sum equal to zero.

20 Find the number of the terms necessary to be taken off from the sequence (27 , 24, 21 , ...) starting from the first term to vanish the sum.

21 **Saves:** Zyad saves 15 L.E from his daily work. If he saves an amount increases for 2 L.E every day more than the day before directly, find how much he can save within 15 days.

22 **School stage:** A school stage accommodates for 16 rows. If the first row contains 16 seats and each row next to it contains a number of seats more than the directly previous row for 4 seats, find the number of seats in this stage.

23 **Food:** Kareem has a store for food commodities. He arranges 78 tuna cans in rows so that he puts 12 cans in the (first) lowest row , 11 in the next row and 10 in the next and so on.. till the last row on which he puts one tuna can. Find the number of the rows .

24 **Loan:** A man has borrowed an amount of money, he has handled to pay it back over 8 installments. How much is the loan if the first installment is 500 L.E and each next installment is more than the directly previous installment for 200 L.E?.

25 **Maintenance:** A company has assigned one of its buildings to be completely maintained and it has stated a date to receive the building. There was a condition in the maintenance contract stating that ; in case of delaying the date of receiving the building, the contractor will pay 100 L.E as a fine for the first day and this fine increases 100 L.E more than the day before . How much will the contractor pay if he delays to receive the building for 6 days?

Activity

Sports: Zyad is preparing himself for a long-distance race. He has decided to run 4 km on the first day, then increased the distance for half a kilometer daily.

1 Find the distance , which Zyad covers on the seventh day.

2 Find the sum of the distances, which Zyad covers in the first week (A week is seven days).

3 How many days does Zyad need to cover 81 km if he continues to train with respect to this pattern?

Unit one

1 - 4 Geometric sequence

You will learn

- Definition of the geometric sequence
- Graphical representation of the geometric sequence
- n^{th} term of the geometric sequence
- Identifying the geometric sequence
- Geometric means
- The relation between an arithmetic mean and a geometric mean of two numbers

Key terms

- Geometric Sequence
- n^{th} Term
- Increasing Sequence
- Decreasing Seqence
- Alternating signal Sequence
- Geometrical Mean

Materials

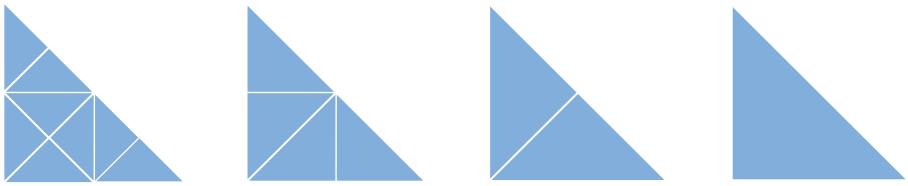
- Scientific calculator
- Graphic program

Activity

(1) On cardboard, draw an isosceles right - angled triangle.

(2) Cut the constructed triangle into two isosceles right- angled triangles.

(3) Repeat the work as shown below and find the number of the triangles resulted each time .



(4) Answer the following question :

a Does the number of the resulted small triangles construct an arithmetic sequence? Explain.

b Is there a relation relating the numbers of the resulted sequence? What is this relation?

c Can you find the number of the triangles resulted in fifth and sixth figures by repeating the pattern above?

From the previous activity, we deduce that :

The sequence resulted from the previous figures is(1 , 2 ,4 , 8 , ...). It is not an arithmetic sequence because $T_{r+1} - T_r \neq$ a constant, but we notice that if any term is divided by the directly previous term, it gives a constant (it is the number 2) . This sequence is called a geometric sequence.

Definition 1

➢ The sequence (T_n) where $T_n \neq 0$ is called a geometric sequence If $\frac{T_{n+1}}{T_n}$ = a constant for each $n \in Z_+$ and the constant is called the common ratio of the sequence and is denoted by the symbol(r).

Example

1 Show which of the following sequences (T_n) is geometric , then find the common ratio of each:

a $T_n = 2 \times 3^n$ b $T_n = 4\, n^2$

c The sequence (T_n) where : $T_1 = 12$, $T_n = \frac{1}{4} \times T_{n-1}$ (where $n > 1$)

Solution

a $\because \frac{T_{n+1}}{T_n} = \frac{2 \times 3^{n+1}}{2 \times 3^n} = 3^{n+1-n} = 3$ **(constant)**

$\therefore$ the sequence is geometric and its common ratio r = 3

b $\because \frac{T_{n+1}}{T_n} = \frac{4(n+1)^2}{4n^2}$ **(is not constant)**

$\therefore$ The sequence is not geometric

c $\because (T_n) = (\frac{1}{4} \times T_{n-1})$ (where $n > 1$)

$\therefore \frac{T_n}{T_{n-1}} = \frac{1}{4}$ **(constant)**

$\therefore$ The sequence is geometric and its common ratio $r = \frac{1}{4}$

Try to solve

1 Tell which of the following sequences is geometric , then find its common ratio in case it is a geometric sequence :

a $(T_n) = (3 , 6 , 12 , 24 , 48 , 96)$

b $(T_n) = (\frac{1}{243} , \frac{1}{81} , \frac{1}{27} , \frac{1}{9} , \frac{1}{3})$

c $(T_n) = (5 \times 2^n)$

d $(T_n) = (3(n+1)^2)$

Graphical representation of geometrical sequence

Example

2 Find the next four terms of the geometric sequence (8, 4, 2, ...), then represent the first seven terms graphically .

Solution

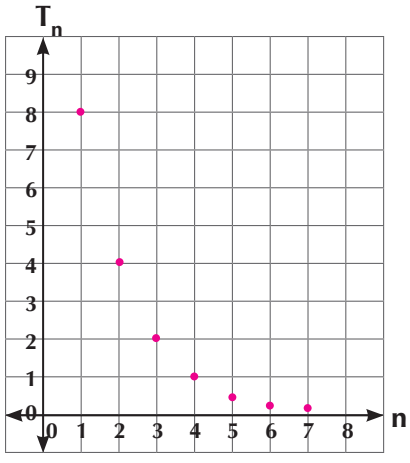
The sequence common ratio = $\frac{1}{2}$, thus, the next four terms are : $1 , \frac{1}{2} , \frac{1}{4} , \frac{1}{8}$

the domain is $\{ 1 , 2 , 3 , 4 , 5 , 6 , 7 \}$

the range is $\{ 8 , 4 , 2 , 1 , \frac{1}{2} , \frac{1}{4} , \frac{1}{8} \}$

From the graph , we notice that:

- The terms of the sequence are decreasing where $a > 0 , 0 < r < 1$
- The graphical representation of the geometric sequence follows the exponential function.



Critical thinking:

Can the common ratio of the geometric sequence be equal to zero or equal to one? Explain.

Try to solve

2 Find the next four terms of the geometric sequence $(\frac{1}{27} , \frac{1}{9} , \frac{1}{3} , ...)$, then represent the first seven terms graphically.

Finding the n^{th} term of a geometrical sequence

From definition (1), the n^{th} term of the geometric sequence (T_n) whose first term is a and common ratio is r can be deduced as follows:

$T_1 = a$, $T_2 = ar$, $T_3 = ar^2$ and $T_4 = ar^3$

By continuing this pattern, we find that the n^{th} term of this sequence is: : $T_n = ar^{n-1}$

Example

(3) In the geometric sequence (2 , 4 , 8 , ...), find:

first : The fifth term. **second:** the order of the term whose value is 512.

Solution

$\because$ $a = 2$, $r = \frac{4}{2} = 2$, $T_n = a \times (2)^{n-1}$

$\therefore$ $T_5 = ar^4 = 2 \times 2^4 = 2 \times 16 = 32$ **i.e. the value of the fifth term is 32**

$\because$ $T_n = a \times r^{n-1}$ $\therefore$ $2 \times 2^{n-1} = 512$ **By dividing the two sides by 2**

$\therefore$ $2^{n-1} = 2^8$ $\therefore$ $n - 1 = 8$ $\therefore n = 9$

i.e. the term whose value is 512 is the ninth term.

Try to solve

(3) Prove that the sequence (T_n) where $T_n = 2 \times 3^{n-5}$ is a geometric sequence, then find its seventh term.

Identifying the geometrical sequence

The geometric sequence can be identified whenever its first term and common ratio are known (given).

Example

(4) (T_n) is a geometric sequence, if $T_4 = 40$ and $T_7 = 320$, find this sequence.

Solution

$\because$ $T_4 = a r^3$ $\therefore$ $a r^3 = 40$ (1)

$\because$ $T_7 = a r^6$ $\therefore$ $a r^6 = 320$ (2)

By dividing the two sides of the equations (1) and (2)

$\therefore$ $\frac{a r^6}{a r^3} = \frac{320}{40}$ (where $a r \neq 0$) $\therefore r^3 = 8$

$\therefore r = 2$

By substituting in equation (1)

$\therefore a (2)^3 = 40$ i.e. : $8 a = 40$

By dividing the two sides by 8, then $a = 5$ $\therefore$ the sequence is (5 , 10 , 20 , ...)

Verbal expression :

What do you expect if the power of the common ratio r is an even number? Explain .

Using the scientific calculator to write the geometric sequence:

To write down the geometric sequence in which a = 5 and r = 2, we do the following:
We write the value of a (number 5), then press [=] then press [×] and put the value of r (number 2), then press [=] then we get the second term of the sequence. By pressing [=] repeatedly, we get the next terms and so on....

Try to solve

4 (T_n) is a geometric sequence in which $T_3 = 12$ and $T_8 = 384$. find this sequence

5 Find the geometric sequence whose terms are positive, its second term equals 6 and its tenth term equals 1536 .

Example

Education:

5 The increasing rate of second secondary grade in an educational administration is 4 % yearly more than the directly previous year. How many students will be there after 6 years if the number of the students is 2400 students right now?

Solution

∵ Number of students now = 2400

∴ Number of students in the second year $= 2400 + 2400 \times 4\ \%$

$= 2400\ (1 + 0.04)$

$= 2400\ (1.04)$

Remember

$4\% = \frac{4}{100} = 0.04$

Number of students in the third year $= 2400\ (1.04) + 2400(1.04) \times 0.04$

$= 2400\ (1.04)\ (1 + 0.04) = 2400(1.04)^2$ and so on ...

i.e. the numbers of students construct a geometric sequence

$(2400\ ,\ 2400(1.04)\ ,\ 2400(1.04)^2, \ldots\)$

$a = 2400 \quad , \quad r = 1.04 \quad , \quad n = 6$

by substituting in the rule of n^{th} term of the geometric sequence $T_n = a \times r^{n-1}$

$T_n = (2400) \times (1.04)^5 = 2919.966966$

i.e. the number of students after 6 years equals 2920 students approximately .

Try to solve

Physical fitness:

6 Kamal goes walking to keep fit and loses weight. If he walks a distance of 50 meters on first day, and on each next day, he walks a distance of twice the directly previous day .

a Write down the sequence of distances which he walks daily.

b Find the distance which he walks on the seventh day.

Geometric Means

The geometric means are similar to the arithmetic means. They are the terms located between two non- successive terms in the geometric sequence . the common ratio of the geometric sequence is used to find these means.

Definition 2

➢ **If a, b and c are three successive terms of a geometric sequence, then b is known as the geometric mean between the two numbers a and c where:** $\frac{b}{a} = \frac{c}{b}$ **i.e.** $b^2 = a\,c$, **then** $b = \pm\sqrt{a\,c}$

Tip

the geometric mean of a set of positive real values $T_1, T_2, T_3, \ldots, T_n$ is known as the n^{th} root of the product of these values. I.e. the geometric mean = $\sqrt[n]{T_1 \times T_2 \times \ldots \times T_n}$

Finding the geometric means :

Example

(6) Insert 5 geometric means between 4 and 2916

Solution

First : Find the number of the sequence terms

There are five means between the first and last terms in the geometric sequence

Thus , the number of the terms of the sequence is $n = 2 + 5 = 7$

Second : Find the value of d

The n^{th} term of the arithmetic sequence : $T_n = a\,r^{n-1}$

By substituting : $a = 4$, $T_n = 2916$, $n = 7$

$2916 = 4 \times r^{7-1}$ i.e. : $4 \times r^6 = 2916$

by dividing the two sides by 4 : $r^6 = 729$ i.e. : $r^6 = (\pm 3)^6$ then $r = \pm 3$

Third : Use the value of r **to find the geometric means required :**

4 , 12 , 36 , 108 , 324 , 972 , 2916 or

×3 , ×3 , ×3 , ×3 , ×3 , ×3

4 , -12 , 36 , -108 , 324 , -972 , 2916

× -3 , × -3 , × -3 , × -3 , × -3 , × -3

the required means are: 12 , 36 , 108 , 324 , 972 or - 12 , 36 , - 108 , 324 , - 972

Try to solve

(7) Insert six geometric means between $\frac{1}{4}$ and 32

Critical thinking: What do you expect about the relation between the arithmetic and geometric means of two equal positive real numbers?

Example

(7) Two positive numbers of arithmetic mean equals 10 and geometric mean equals 8 . Find the two numbers.

Solution

Let the two numbers be x and y

Definition of the arithmetic mean : $\frac{x + y}{2} = 10$

By multiplying the two sides of the equation by 2 : $x + y = 20$, then : $y = 20 - x$ **(1)**

Definition of the geometric mean : $\sqrt{x\,y} = 8$

By squaring the two sides : $x\,y = 64$ **(2)**

By substituting from equation **(1)** in equation **(2)**: $x(20 - x) = 64$

By expanding the brackets and simplifying : $x^2 - 20x + 64 = 0$

By factorizing the trinomial : $(x - 4)(x - 16) = 0$

Solving the quadratic equation : $x = 4$ or $x = 16$

By substituting to find the two values of y : $y = 16$ or $y = 4$

i.e. the two numbers are 4 and 16

Try to solve

(8) Two positive numbers of arithmetic mean equals 25 and geometric mean equals 20. Find the two numbers.

Example

(8) If you insert four geometric means between two numbers and the sum of the first and fourth means equals 90 and the sum of the second and third means equals 60, find the two numbers.

Solution

$\because$ Number of means =4 $\therefore$ number of sequence terms = 4 + 2 = 6

$\therefore$ let the first number = a, then the first and fourth terms are T_2 and T_5

$\therefore T_2 + T_5 = 90$ $\therefore a r + a r^4 = 90$

$\therefore a r (1 + r^3) = 90$ **(1)**

$\because$ the second and third terms are T_2 and T_4

$\therefore a r^2 + a r^3 = 60$ $\therefore a r^2 (1 + r) = 60$ **(2)**

By dividing **(1)** , **(2)** $\frac{ar(1+r)(a - r + r^2)}{ar^2(1 + r)} = \frac{3}{2}$

$\therefore 2r^2 - 2r + 2 = 3r$ $\therefore 2r^2 - 5r + 2 = 0$ $\therefore (r - 2)(2r - 1) = 0$

$\therefore r = 2$ or $r = \frac{1}{2}$

By substituting r = 2 $\quad\therefore$ a = 5

By substituting r = $\frac{1}{2}$ $\quad\therefore$ a = 160 $\quad$ the two numbers are: 160 , 5

Try to solve

(9) If several geometric means are inserted between 2 and 1458 and the ratio between the sum of the first two means and the last two means is 1 : 27, find the number of these means.

Exercises 1 - 4

Determine the geometric sequences in the following , then find the common ratio in case the sequence is geometric:

(1) (1 , 4 , 9 , 16 , ...)

(2) (243 , 81 , 27 , 9 , ...)

(3) ($\frac{1}{128}$, $\frac{1}{64}$, $\frac{1}{32}$, $\frac{1}{16}$, ...)

(4) $\left(\frac{1}{2^n}\right)$

(5) (- 1 , 3 , - 9 , - 27 , ...)

(6) ($n^2 - 1$)

Write down the first five terms of the geometric sequence, if known that :

(7) a = 2 , r = 4

(8) a = -4 , r = 2

(9) a = 1 , r = $\frac{-1}{2}$

(10) a = -128 , r = $\frac{-1}{2}$

Complete

(11) The seventh term of the geometric sequence (64 , 32 , 16 , ...) equals

(12) The sixth term of the geometric sequence ($\frac{1}{243}$, $\frac{1}{81}$, $\frac{1}{27}$, ...) is

(13) The fifth term of the sequence (T_n) where $T_n = 2 \times (3)^{n-1}$ equals

(14) The n^{th} term of the geometric sequence (3 , - 6 , 12 , ...) is

(15) The geometric mean of the two numbers 4 and 16 is

(16) If the geometric mean of the two numbers 9 and y is 15 , then y equals

(17) If a , b and c are three consecutive and positive terms of a geometric sequence, then b = ...

Choose the correct answer:

(18) The fifth term of the geometric sequence (8 , 6 , $\frac{9}{2}$, ...) is :

a) $\frac{27}{8}$ $\quad$ b) $\frac{27}{16}$ $\quad$ c) $\frac{9}{4}$ $\quad$ d) $\frac{81}{32}$

(19) All the following sequence are geometric except :

a) (3 , -6 , 12 , -24 , ...) $\quad$ b) (-2, 4, 10, 16, ...)

c) ($\frac{3}{2}$, 1 , $\frac{2}{3}$, $\frac{4}{9}$) $\quad$ d) ($\frac{3b}{a}$, 6 , $\frac{12a}{b}$, $\frac{24a^2}{b^2}$, ...) where a > 0 , b > 0

(20) Of all the following sequences , the geometric sequence is :

a) $(T_n) = (4n^2)$ for each $n \geq 1$ $\quad$ b) $(T_n) = (\frac{1}{4} \times T_{n-1})$ for each $n \geq 2$

c) $(T_n) = (2^n - 1)$ for each $n \geq 1$

21 If a , b and c are three consecutive positive terms of a geometric sequence , then :

a $a\,c = b^2$ b $a\,c = 2\,b$ c $\frac{a+c}{2} = b$ d $b^2 = a + c$

Discover the error:

22 The terms of the geometric sequence are represented by a set of separated points which are collinear.

23 The sequence (T_n) is called geometric if $\frac{T_n}{T_{n+1}}$ equals a constant known as the common ratio of the sequence (for each $n \geqslant 1$).

24 The geometric means are the terms lying between two consecutive terms of a geometric sequence and can be found whenever the number of these means is known.

25 The arithmetic mean of two different real numbers is greater than their geometric mean.

Answer the following questions:

26 If (T_n) is a sequence where $T_n = 5 \times 2^n$, prove that it is a geometric sequence, then write its first three terms.

27 In the geometric sequence $\left(\frac{1}{8}, -\frac{1}{4}, \frac{1}{2}, -1, ...\right)$, find :

a Its tenth term b The order of the term whose value = - 1024 .

28 Show that the sequence (T_n) where $T_n = \frac{3}{8}(2)^n$ is a geometric sequence , then find its eighth term and the order of the term whose value is 768.

29 Find the geometric sequence whose common ratio = $\frac{1}{2}$ and its third term = 24 .

30 Find the geometric sequence whose first term = 9 and its sixth term = 288 .

31 Find the geometric sequence (T_n) in which $T_3 = 12$ and $T_8 = 384$.

32 Find the geometric sequence whose third term = 18 , and its sixth term = 486

33 Find the geometric sequence (T_n) in which $T_2 = 10$ and $T_6 = 160$.

34 Find the geometric mean between 16 and 49

35 Find the two numbers whose arithmetic mean is 5 and geometric mean is 3.

36 Find two positive numbers whose positive geometric mean is greater than one of those two numbers by 2 and is less than the other number by 3. .

37 Insert four geometric means between 2 and 64 .

38 Insert five positive geometric means between $\frac{8}{27}$, $\frac{27}{8}$

39 If the geometric mean of the two numbers 2 and x is 8, find the value of x.

40 Two positive numbers , the difference between them is 8 and their geometric mean is 3, find the two numbers.

41 **Environment:** Water is poured in a water tank at a rate of twice the directly previous day. How many days does the water tank need to hold 1536 liters if 12 liters has been poured in the tank on the first day?

42 **Population :** The population of a city increases at a constant rate of 3% per year. How many populations are there in this city after 5 years if the populations are 600000 at present?

43 **Trade:** The price of a new brand car is 120000 L.E. How much is that car after 5 years if its price decreases at a rate of 12% from the directly previous year?

44 **Income:** The salary of an employee is 1200 L.E and he gets a yearly raise at a rate of 6% more than his salary in the directly previous year. How much is his salary after 6 years?

Activity

Overpopulation

If the populations of a city is 80000 and it is known that the rate of population increasing each year 3% more than the directly previous year .



a Find the number of populations after one year.

b Find the number of populations after two years.

c Write down the geometric sequence representing the populations of this city .

d How many populations will be there at the end of the tenth year?

e Use the (*Excel program*) to represent these data at the end of five years .

f Log in internet and do a research about the overpopulation in Egypt, how badly it affects the national income , and the solutions suggested to control this problem.

Geometric Series

1 - 5

You have previously learned that the series is the sum of a sequence terms and you also learned how to find the sum of an arithmetic series. Can you find the sum of the following geometric series?

$95 + 285 + 855 + \ldots\ldots\ldots\ldots 1869885$. Notice that it is quite difficult to find the sum using the traditional way. As a result, we are in need to a rule to find this sum easily and quickly. This is what we are going to know now .

The sum of geometric series

The geometric series is the sum of the terms of the geometric sequence between which the addition sign (+) is placed and the sum of its terms is denoted by the symbol S.

Sum of first n terms of a geometric sequence

First : Finding the sum of n terms of a geometric series in terms of its first term and the common ratio

If $a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ is a geometric series whose first term is a and common ratio is r, then the sum S_n of this series can be found as follows :

$s_n = a + ar + ar^2 + \ldots\ldots + ar^{n-2} + ar^{n-1}$ (1)

By multiplying the two sides by r, then :

$rs_n = ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} + ar^n$ (2)

By subtracting the two equations :

$s_n - rs_n = a - ar^n$ i.e. :

$s_n (1 - r) = a(1 - r^n)$

By dividing the two sides by $(1 - r)$ in a condition $(1 - r) \neq 0$

$$s_n = \frac{a(1 - r^n)}{1 - r} \quad , r \neq 1$$

Critial thinking :

what is the value of the sum when r = 1 ?

Example

1 Find the sum of the geometric series in which : a = 3 , r = 2 and n = 8

You will learn

- Sum of the geometric series.
- Using summation notation
- Infinite geometric series
- Sum of infinite geometric series.
- Converting the recurring decimal into an irrational number .

Key terms

- Geometric Series
- Infinite Geometric Series

Materials

- Scientific calculator
- Graphical programs

Solution

the sum of the geometric sequence : $s_n = \frac{a(1 - r^n)}{1 - r}$

by substituting : $a = 3$, $r = 2$, $n = 8$

$s_8 = \frac{3(1 - 2^8)}{1 - 2}$ by simplifying $s_8 = 3 \times 255 = 765$

Try to solve

1 Find the sum of the following two geometric sequences in which:

a $a = 4$, $r = 3$, $n = 6$ b $a = 1000$, $r = \frac{1}{2}$, $n = 10$

Second: finding the sum of n terms of a geometric series in terms of its first and last terms

We know that : $s_n = \frac{a - ar^n}{1 - r}$ (1)

and : $\ell = a\, r^{n-1}$ by multiplying the two sides by r then $\ell r = a\, r^n$ (2)

by substituting from (2) in (1), then : $s_n = \frac{a - \ell r}{1 - r}$, $r \neq 1$

Example

2 Find the sum of the geometric series: $1 + 3 + 9 + \ldots\ldots + 6561$

Solution

The sum formula of the geometric sequence : $s_n = \frac{a - \ell r}{1 - r}$

By substituting : $a = 1$, $r = 3$, $\ell = 6561$

$s = \frac{1 - 6561 \times 3}{1 - 3}$ by simplifying $s = \frac{19682}{2} = 9841$

Try to solve

2 Find the sum of the geometric sequence in which: $a = 2048$, $r = \frac{1}{2}$ and $\ell = 28$

Using the Summation Notation.

Example

3 Find : $\sum_{r=5}^{12} 3(2)^{r-1}$

Solution

As the expression inside the summation notation is in the exponential form, it represents a geometric sequence

$T_5 = a = 3(2)^{5-1} = 48$, $r = 2$, $n = 12 - 5 + 1 = 8$

the sum of the geometric series : $s_n = \frac{a(1 - r^n)}{1 - r}$

by substituting : $a = 48$, $r = 2$, $n = 8$

$s_8 = \frac{48\,(1 - 2^8)}{1 - 2}$ **by simplifying** $s_8 = 48 \times 255 = 12240$

Think :

Can you find the sum in the previous example in terms of a , ℓ and r ? Explain.

Try to solve

3 Find the sum of the following two series :

a $\sum_{r=7}^{16} \frac{1}{8}(2)^{r-1}$ b $\sum_{r=3}^{11} 16(\frac{1}{2})^{r-1}$

Example

Forming the geometric sequence

4 Find the geometric sequence whose first term = 243, last term = 1 , and the sum of its terms is 364.

Solution

$\because$ a = 243 , ℓ = 1 , s_n = 364 , $s_n = \frac{a - \ell r}{1 - r}$

$\therefore$ $364 = \frac{243 - r}{1 - r}$ $\therefore$ $364(1 - r) = 243 - r$

$\therefore$ $364 - 364r = 243 - r$ $\therefore$ $364r - r = 364 - 243$

$\therefore$ $363r = 121$ by dividing the two sides by 363 $\therefore$ $r = \frac{1}{3}$

The geometric sequence is (243 , 81 , 27 , ...)

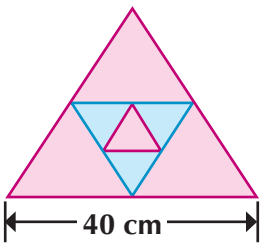
Try to solve

4 Find the geometric sequence whose first term is 243, last term is 1, and the sum of its terms is 364.

Example

Geometry:

5 The opposite figure shows an equilateral triangle whose side length is twice the side length of the interior triangle which its vertices are the mid-points of the exterior triangle sides. If this pattern continues interiorly, find the sum of the perimeters of the first 10 triangles in this pattern.



Solution

The perimeter of the greatest triangle = $3 \times 40 = 120$ cm

The perimeter of the next smaller triangle = $3 \times 20 = 60$ cm

The perimeter of the triangle next to the smaller triangle = $3 \times 10 = 30$cm

i.e. the pattern is: 120 , 60 , 30 , ... to 10 terms

Sum of perimeters = 120 + 60 + 30 + ... it is the sum of a geometric series

The sum of the geometric series : $s_n = \frac{a(1 - r^n)}{1 - r}$

By substituting : a = 120 , $r = \frac{1}{2}$, n = 10

By simplifying and using the calculator $S_{10} = \frac{120(1 - (\frac{1}{2})^{10})}{1 - \frac{1}{2}} = 239\frac{49}{64}$ cm

Remember

Perimeter of equilateral triangle = 3 × its side length

Try to solve

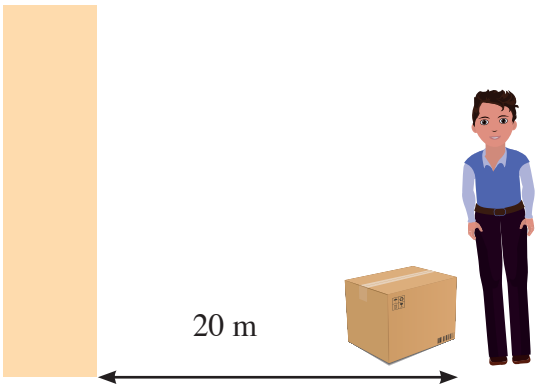
5 **Biology:** Culturing bacteria is replicated every day (in a medical research sample) . How many bacteria are there after ten days if their number is 800 on the first day ?

Infinite geometric series

Think and discuss

Zyad has wanted to move a box in the direction of a wall distant 20 m from him over some stages so that the distance travelled by the box equals half the remaining distance after each stage. Can Zyad reach the wall?

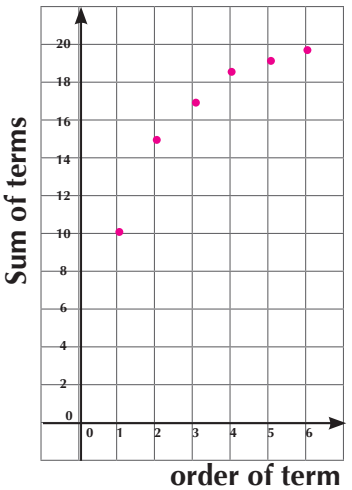
You can answer this question through learning the infinite geometric series .



Definition 2

The infinite geometric series has an infinite number of terms. If their sum is a real number, the series is convergent because its sum gets near to a real number . If the series doesnot have a sum , it is divergent.

In Think and discuss, the sum of the distances traveled by Zyad is given by the series 10 + 5 + 2,5 +1,25 + ... the more its terms increase, the nearer its sum gets to 20 m. It is the real sum of the series. As a result , we can assume that Zyad will reach the wall when the terms of the sequence increase infinitely. The figure opposite illustrates the graphical representation of the sum of S_n , so the sum of the convergent series gets near to a real number where $|r| < 1$ and the series is divergent if the sum does not get near to a real number where $|r| \geqslant 1$



Example **Convergent and divergent series**

6 Which of the following geometric series can an infinite number of its terms be summed? Explain.

a 75 + 45 + 27 + ...

b 24 + 36 + 54 + ...

Solution

a Find the common ratio of the geometric series $r = \frac{45}{75} = \frac{3}{5}$, then an infinite numbers of terms of the series can be summed because $-1 < \frac{3}{5} < 1$

b Find the common ratio of the geometric series $r = \frac{36}{24} = \frac{3}{2}$, then the infinite number of terms of the series can not be summed because: $\frac{3}{2} > 1$

Try to solve

6 Which of the following geometric series can you sum an infinite number of its terms? Explain

a $7 + 21 + 63 + ...$ **b** $\frac{81}{8} + \frac{27}{4} + \frac{9}{2} + ...$

The sum of the infinite geometric series

We knew that the sum of n terms of the geometric series is given by the relation $S_n = \frac{a(1 - r^n)}{1 - r}$

and when we sum an infinite number of its terms, then r^n gets near to zero when $-1 < r < 1$

and the sum became : $s_\infty = \frac{a}{1 - r}$

Critical thinking:

Can the sum of an infinite geometric series be found when $|r| \geqslant 1$? Explain

Verbal expression :

Can you find the sum of the geometric series in **Think and discuss**? Explain.

Example

7 Find the sum for each of the following two geometric series if found :

a $\frac{81}{8} + \frac{27}{4} + \frac{9}{2} + ...$ **b** $\frac{2}{3} + \frac{5}{6} + \frac{25}{24} + ...$

Solution

a Find the common ratio of the geometric sequence : $r = \frac{27}{4} \div \frac{81}{8} = \frac{27}{4} \times \frac{8}{81} = \frac{2}{3}$

$\because -1 < \frac{2}{3} < 1$ $\therefore$ The series has a sum

$\because a = \frac{81}{8}$, $r = \frac{2}{3}$ **by substituting in the sum formula** $s_\infty = \frac{a}{1 - r}$

$\therefore s_\infty = \frac{\frac{81}{8}}{1 - \frac{2}{3}} = \frac{\frac{81}{8}}{\frac{1}{3}} = \frac{243}{8}$

b Find the common ratio of the geometric sequence : $r = \frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \times \frac{3}{2} = \frac{5}{4}$

$\because \frac{5}{4} > 1$ $\therefore$ The series is divergent and has no sum

Try to solve

7 Find the sum for each of the following two geometric series if found :

a $12 - 24 + 48 - 96$ **b** $\frac{7}{5} + \frac{21}{10} + \frac{63}{20} + ...$

Example **Using the summation notation**

8 Find $\sum_{r=1}^{\infty} 42(\frac{6}{7})^{r-1}$

Solution

Sum of the geometric sequence : $s_\infty = \frac{a}{1 - r}$

by substituting : $a = 42$ and $r = \frac{6}{7}$: $s_\infty = \frac{42}{1 - \frac{6}{7}} = 294$

Try to solve

8 Find : $\sum_{r=1}^{\infty} 56(\frac{3}{4})^{r-1}$

Applications on the Sequences

Sequences can be used to study some life problems related to humans such as income resources and the changes happen to people like reproduction, death and immigration. The following examples show that .

Example

9 **Income:** The following table represents the salary of an employee at the end of five consecutive years in L.E known that this income is subjected to a geometric sequence .

Year	first	second	third	fourth	fifth
Salary in L.E	288	432	648		

a Find the common ratio of the geometric sequence .

b Find the salary of that employee within the fourth and fifth years .

c Find the sum of what this employee deserves during this period in L.E .

d Find the monthly average income of this employee during this period .

Solution

a ∵ The sequence is geometric ∴ The common ratio (r) $= \frac{\text{second term}}{\text{first term}} = \frac{432}{288} = \frac{3}{2}$

b The monthly income of the fourth year $= T_3 \times r = 648 \times \frac{3}{2} = 962$ L.E ,

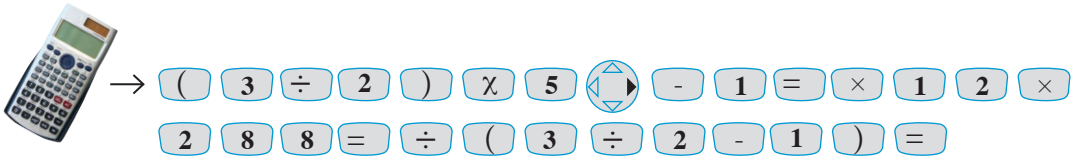
The monthly income of the fifth year $= T_4 \times r = 962 \times \frac{3}{2} = 1458$ L.E

c The sum of what the employee deserves = 12 (288 + 432 + 648 + ... to five terms)

∵ $s_n = \frac{a(1 - r^n)}{1 - r}$ ∴ $s_n = \frac{288 \times 12\,[1 - (\frac{3}{2})^5]}{1 - \frac{3}{2}}$

$= 45576$ L.E

Use the arrangement of the calculator buttons as follows :



d The monthly average of the employee's income = $\frac{\text{sum of income}}{\text{number of months}} = \frac{45576}{5 \times 12} = 759.6$ L.E

Critical thinking: Can you find the monthly average of the employee's income by dividing (288 + 432 + 648 + ... to 5 terms) by 5 years ? Explain.

Try to solve

Physics: A small ball of iron has been rolled over on a horizontal plane. If the ball covered 25m in the first minute , then it started to cover 60% only in each next minute from the distance it covered in the directly previous minute. Find the total distance which the ball covered till it stopped completely.

Exercises 1 - 5

Choose the correct answer from those given:

1. The sum of the first five terms of the geometric sequence in which a = 1 and r = 2 equals :

 a 32 **b** 31 **c** 30 **d** 29

2. The sum of an infinite number of terms of the sequence (4 , 2 , 1 , ...) is :

 a 8 **b** 12 **c** 16 **d** 20

3. If the sum of an infinite number of terms of a geometric sequence whose common ratio is $\frac{1}{4}$ equals 4 , then its first term equals :

 a 1 **b** 2 **c** 3 **d** 4

4. If the sum of an infinite number of terms of the geometric sequence whose first term is 12 equals 96, then its common ratio equals :

 a $\frac{1}{3}$ **b** $\frac{1}{2}$ **c** $\frac{7}{8}$ **d** $\frac{3}{4}$

5. If the first term of a geometric sequence equals the sum of the next terms to infinity, then the common ratio of this sequence equals :

 a 0.5 **b** 0.333 **c** 0.25 **d** 0.666

Discover the error:

6. The sum of an infinite geometric series can be found when $|r| \leqslant -1$

7. The sum of the first n terms of the geometric sequence in which $|r| < 1$ is greater than the sum of an infinite number of its terms .

8. The sum of an infinite number of terms of the sequence (16 , 8 , 4 , ...) is greater than twice its first term .

Answer the following questions:

9 Find the sum for each of the following geometric sequences :

a (6 , 12 , 24 , ... to 6 terms)

b (125 , 25 , 5 , ... to 6 terms)

c (3 , - 6 , 12 , ... , 768)

10 Which of the following geometric sequences can be summed up to ∞, then find the sum if possible :

a (24 , 12 , 6 , ...)

b (3 , - 6 , 12 , ...)

c $(\frac{1}{32}, \frac{1}{16}, \frac{1}{8}$...)

d $(2 \times 5^{1-n})$

11 Find the sum of an infinite number of terms for each of the following geometric sequence :

a (32 , 16 , 8 , ...)

b $(\frac{81}{16}, \frac{27}{8}, \frac{9}{4}$)

c $(2, \sqrt{2}, 1,$)

d $(\frac{27}{4}, \frac{27}{16}, \frac{27}{64}$)

d $(T_n) = (3^{3-n})$

e $(T_n) = (\frac{2}{3})^{n-1}$

12 Find the geometric sequence whose first term = 243 , last term = 1 , and the sum of its terms = 364

13 Find the geometric sequence whose sum is 1093, last term is 729 and its common ratio is 3

14 Find the sum of the terms of the geometric sequence $(T_n) = (3^{n-1})$ starting from its fourth term up to tenth term.

15 In the geometric sequence (1 , 3 , 9 , ...), find the least number of terms that should be taken off starting from its first term to make the sum of these terms greater than 1000

16 Find the geometric sequence whose sum of an infinite number of its terms equals 48 and its second term equals 12 .

17 How many terms need to be taken off from the geometric sequence (3 , 6 , 12 , ...) starting from its first term to make the sum of these terms = 381.

18 Prove that the sequence $(T_n) = (10 \times 2^{n-2})$ is a geometric sequence , then find the number of the terms starting from the first term to make the sum 2555 .

19 (T_n) is a geometric sequence whose terms are positive $T_2 = 6$, $T_3 - T_1 = 9$. find the sequence and sum of the first twelve terms .

20 Find the geometric sequence whose terms are positive , the sum of its first four terms = 45 , and its sixth term is greater than its second term by 90 .

21 If the first term of a geometric sequence of an infinite number of terms = 18 , and its fourth term = $\frac{16}{3}$, find its sum.

22 Find the geometric sequence whose sum of its first and second terms = 16 and the sum of an infinite number of its terms = 25

23 An infinite geometric sequence whose first term = the sum of next terms to infinity and the sum of the first and second terms = 9. Find the sequence .

24 An infinite geometric sequence in which any term of it = twice the sum of the next terms to infinity. If its fourth term = 3 , find this sequence.

25 **Income:** A worker has started to work at a factory for a yearly income of 7200 L.E so that he got a yearly raise of 0.6 % of the directly previous year. Calculate his income in the seventh year and the sum of what he got within the first seven years. .

26 **Income :** An employee has started to work for a yearly income of 3600 and this income increases each year at a rate of $\frac{1}{20}$ more than the directly previous year. How much will he have after 11 years? What is the sum of money which this employee receives during this period of time?

27 **Saving :** At the age of 6, Kreem could save 150 L.E. in the year and in each directly next year, he could save twice of what he saved in the directly previous year. He is 10 years-old now and wants to buy a computer for 5000 L.E. Is the sum of what he saved during this period enough to buy this computer? Explain .

28 **Critical thinking :** If a , b and c form a geometric sequence, prove that : $\frac{a^2 + b^2}{b^2 + c^2} = \frac{a^2}{b^2}$

29 **Sports:** A rubber ball fell from a height of 36 m, and in each time it hits the ground , it rebounces back $\frac{3}{4}$ the distance it fell from . Find:

a The distance from which the ball fell to hit the ground in the sixth time .

b Sum of distances which the ball covered from the time of its falling to the time it hit the ground in the sixth time .

c The sum of distances which the ball covered from the time of falling until to be at rest.

Unit summary

Sequence: It is a function whose domain is the set of positive integers Z_+ or a subset of it and its range is a set of real numbers R with regarding the follows:

- Terms of the sequence are the image of its domain elements .
- The symbol (T_n) expresses the sequence whereas symbol T_n express its n^{th} term.
- The sequence is subjected to the order of its elements and these elements can be counted.
- The sequence is finite if the number of its terms is finite and the sequence is infinite if the number of its terms is infinite.
- The sequence is represented graphically as a function throughout representing the ordered pairs by separated points on the coordinate plane .

Series: it is the sum of the sequence terms. The summation notation " Σ " is used to write the series in a short form .

- **Finite series:** it contains a finite number of elements and it is written in the form $\sum_{r=1}^{n} (T_r)$
- **Infinite series:** its terms are infinite and it's written in the form $\sum_{r=1}^{\infty} (T_r)$

Arithmetic sequence: it is the sequence in which the difference between each term and the directly previous term equals a constant called the common difference of the sequence. It is usually denoted by the symbol (d) where $d = T_{n+1} - T_n$ for each $n \in Z^+$

- The arithmetic sequence can be formed in terms of its first term(a) and its common difference (d) .
- The relation between n and T_n is $T_n = dn + b$ where d and b are two constants and d is the sequence common difference. This is a liner relation and it can be graphically represented by a set of points.
- The sequence is increasing if $d > 0$ and is decreasing if $d < 0$
- The n^{th} term of the arithmetic sequence (T_n) whose first term a and common difference d are given by the relation : $T_n = a + (n - 1)\, d$
- To find the last term (l) , then $l = a + (n - 1)\, d$
- If a , b and c are three consecutive terms in an arithmetic sequence, then b is known as the arithmetic mean between the two numbers a and b where $b = \frac{a + c}{2}$
- Several arithmetic means can be inserted between two numbers and the number of the sequence terms = number of means +2.

- The sum of the first n terms of an arithmetic series:

 First : in terms of its first term and the common difference : $s_n = \frac{n}{2}[2a + (n - 1)d]$

 Second : in terms of its first and last term : $S_n = \frac{n}{2}(a + l)$

Geometric sequence: the sequence (T_n) where $T_n \neq 0$ is called a geometric sequence if $\frac{T_{n+1}}{T_n}$ = constant for each $n \in Z^+$ and the constant is called the common ratio of the sequence and it is denoted by the symbol (r)

- Representing the geometric sequence follows the exponential function throughout ordered pairs with separated points on the coordinate plane.
- The n^{th} term of the geometric sequence (T_n) whose first term a and common ratio r is given by the relation $T_n = a \times r^{n-1}$
- If a , b and c are three consecutive terms of a geometric sequence, then b is known as the geometric mean between the two numbers a and c where $b = \pm\sqrt{ac}$
- The sum of the first n terms of a geometric series :

 First: in terms of its first term and the common ratio : $s_n = \frac{a(1 - r^n)}{1 - r}$ and $r \neq 1$

 Second: in terms of its first and last terms : $s_n = \frac{a - \ell r}{1 - r}$ and $r \neq 1$

- **The infinite geometric sequence :** has an infinite number of terms .
- The sum of the infinite geometric sequence : $s_\infty = \frac{a}{1 - r}$ and $r \neq 1$ where $|r| < 1$

General exercises

Complete:

1. In the sequence (T_n) where $T_{n+1} = n\ T_n$. If $T_1 = 1$, then $T_2 =$
2. The series $5 \times 1 + 5 \times 2 + 5 \times 3 + ... + 5 \times 10$ is written by the summation notation in the form
3. The n^{th} term of the arithmetic sequence $(3 , 7 , 11 , ...)$ is
4. The arithmetic mean of the two numbers 8 and 18 is
5. If the arithmetic mean of the two numbers a and 21 is 14 , then a = ...
6. The number of the integers included between 1 and 99 divisible by five equals
7. The sum of the series $\sum_{r=1}^{10} (2r - 3)$ equals
8. The sum of the series $\sum_{r=1}^{5} (2)^{r-1}$ equals

Choose the correct answer

9. The sequence whose rule is : $\mathbf{T_n} = 4n + 1$ is called a(n) sequence:

 a increasing **b** decreasing **c** alternating **d** constant

10. The arithmetic series: $3 + 7 + 11 + 35$ is written using the summation notation as follows:

 a $\sum_{r=1}^{8} (4r - 1)$ **b** $\sum_{r=1}^{9} (4r - 1)$ **c** $\sum_{r=1}^{9} (3r - 4)$ **d** $\sum_{r=1}^{8} (3r - 4)$

11. All the following sequences are arithmetic except :

 a $(11 , 15 , 19 , ...)$ **b** $(-23 , -28 , -33 , ...)$

 c $(\frac{1}{2} , \frac{1}{3} , \frac{1}{4} , ...)$ **d** $(\frac{32}{5} , \frac{26}{5} , \frac{20}{5} , ...)$

12. If $2a + 2 , 6a - 2$ and $7a$ are three consecutive terms of an arithmetic sequence , then a equals :

 a 1 **b** 2 **c** 3 **d** 4

13. The sum of the arithmetic series $\sum_{r=3}^{8} (2r + 1)$ equals :

 a 64 **b** 72 **c** 76 **d** 80

14. The sum of the first n terms of a geometric sequence is given by the relation $S_n = (3^n - 1)$, then its fifth term equals::

 a 162 **b** 243 **c** 81 **d** 729

15. If the sum of an infinite number of terms of a geometric sequence whose first term is 48 is 72 , then its common ratio equals:

 a $\frac{1}{4}$ **b** $\frac{1}{3}$ **c** $\frac{1}{2}$ **d** $\frac{3}{4}$

Answer the following questions

16 Write down the first six terms for the following two sequences defined as follows :

a $T_{n+1} = n\ T_n$, $T_1 = 3$ for each $n \geqslant 1$

b $T_1 = 2$, $T_2 = 5$, $T_{n+2} = T_{n+1} + T_n$ for each $n \geqslant 1$

17 If the sum of the second and fourth terms of an arithmetic sequence equals 2 and the sum of the sixth, seventh and eighth terms equals- 45, find this sequence .

18 Three numbers are arithmetically sequent whose sum equals 12 and their product = 48, find these numbers .

19 Three numbers form an arithmetic sequence whose sum equals 27 and the square of its second term is greater than the product of the first and third terms by 16, find these numbers.

20 If several arithmetic means are inserted between 3 and 53 and the ratio between the sum of the first two means to the sum of the last two means is 3 : 13 , find these means .

21 In the arithmetic sequence (9 , 12 , 15 , ...). Find:

a Sum of first 15 terms.

b Sum of terms of the sequence starting from the fifth term up to the fifteenth term

c Number of terms whose sum = 750 starting from the first term.

22 In the geometric sequence (2 , 6 , 18 , ...), find the least number of the terms that must be taken off starting from the first term to make the sum of these terms greater than 6500.

23 (T_n) is a geometric sequence of positive terms in which $T_6 = \frac{1}{9} T_4$, $T_3 + T_6 = 28$ find the sequence , then find the sum of its first ten terms.

24 A geometric sequence of positive terms whose sum of the first four terms = 60 , and the sum of the next four terms = 960 . Find this sequence , then find the sum of seven terms starting from its third term.

25 If the first term of a geometric sequence equals 81, and its sixth term= $\frac{1}{6}$, find the sum of an infinite number of its terms starting from its third term .

26 **Salaries:** An employee gets a salary of 1200 L.E in the first year. If his salary increases at a rate of 10% yearly of the directly previous year. Use the summation notation to write the sum of what this employee gets within five years, then find this sum.

Accumulative test

Comparisons: Show the aspects of similarities and differences betwee each of the following**:**

1. Sequences and series.
2. The n^{th} term of the arithmetic sequence and the n^{th} term of the geometric sequence.
3. The sum of the arithmetic series and the sum of the geometric series.

Short-answerd questions :

4. Find the value of the first term (a) in each of the following :
 - **a** The arithmetic sequence in which : $d = 3$, $T_7 = 32$
 - **b** The geometric sequence in which : $r = \frac{1}{2}$, $T_4 = \frac{1}{16}$
 - **c** The arithmetic series in which : $d = 6$, $S_{\text{the first } 12} = 216$
 - **d** The geometric series in which : $r = 2$, $S_{\text{the first } 8} = 255$

Multi-answered question:

5. Write down three sequences so that one of them is arithmetic, the second is geometric and the third is neither arithmetic nor geometric and each one of them starts as follows : 2 , 6 , ...
6. Write down three geometric sequence in which the first term of each is 16 so that the first sequence is increasing , the second is decreasing and the third is constant, then find the sum of one of these sequences to infinity .
7. Write down the geometric series $64 + 32 + 16 + \ldots$ in two different ways using the summation notation.
8. **Justification:** when does the infinite geometric series have a sum and when it doesn't have a sum? Justify.

Life application:

9. **Loan:** A merchant has borrowed an amount of money and agreed to pay it back as follows: The first installment is 5000 L.E and each next installment increases 500 L.E more than the directly previous installment . How much is the loan if the number of the installments is 10?
10. **Income:** If the yearly income of an institution is 840000 L.E . After how many years does the income of the institution become 1464000 L.E if the money increases 48000L.E yearly.

11 **Trade:** A man bought an electrical appliance and paid 800 L.E of its price. He agreed with the sales person to pay the rest of the price in monthly installments for two years. The first installment was 100 L.E and each installment increases 10 L.E after that . Find the sum of money which the man pays for the appliance.

Long-answered questions :

12 If you form (15 , (3 k + 2) , (4 k – 5) , ...) as an arithmetic sequence, then find the value of K .

13 Find the arithmetic sequence whose sum of fifth and tenth terms equals 20 and its seventh term is 3 times its fourth term

14 Find the arithmetic sequence whose twentieth term equals 41 and the sum of its third and sixth terms is greater than its ninth term by a unit.

15 Find the geometric sequence (T_n)in which $T_2 = 5$, $\frac{T_7}{T_4} = \frac{1}{27}$.

16 An arithmetic sequence in which $T_{12} = 38$, $T_{35} = 245$, find the sum of the first fifteen terms.

17 An arithmetic sequence in which the sum of the nineteenth and twentieth terms equals 144 and the sum of the twentith and twenty first terms equals 152, Find:

a The twentith term.

b The number of terms necessary to be taken off from the sequence starting from the first term to make the sum 720

Unit Two



Permutations, Combinations

Unit introduction

Counting is an important part of the basic skills in mathematics. We regularly face a lot of problems that need to be solved. We need to do counting operations in deferent ways to solve them. In this unit, we are going to identify different strategies for counting such as the fundamental counting principle and the most important applications of it:

Permutations. They are used to know the number of methods used to order the elements of a set with all possible methods.

Combinations. They mean to choose disregarding the order.

Scientists such as Omar Alkhaiam, Isac Newton and pascal had played a great role in this field which is still ongoing up to day.

Unit objectives

By the end of the unit and doing the activities involved , the student should be able to:

- Identify the counting principle and simple applications on it.
- Identify an introduction about the permutation and combination and the relation between them .
- Use the computer to calculate each of permutations and combinations.

Key - terms

- Fundamental Counting Principle
- Conditional Counting Principle
- Operation
- Tree Diagram
- Factorial
- Permutations
- Combinations
- Elements
- Order
- Committee
- Subset

Unit lessons

Lesson (2 - 1): Counting Principle .

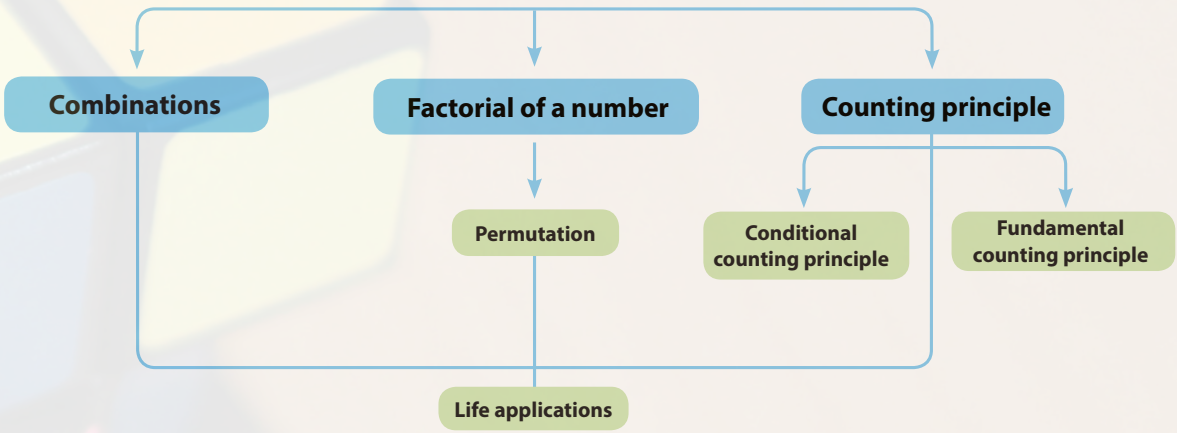
Lesson (2 - 2): Factorial of a number - permutations.

Lesson (2 - 3): Combinations

Materials

Scientific calculator – computer – graphic programs

Unit planning guide



Unit Two

2 - 1 Counting Principle

We will learn

- The concept of counting principle and simple applications on it.
- Fundamental counting principle.
- Conditional counting principle.

Key - term

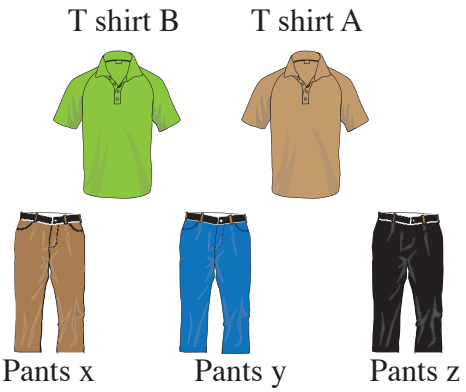
- Fundamental Counting Principle
- Operation
- Tree Diagram

Matrials

- Scientific Calculator
- Graphic program

Think and discuss

How many ways are there to choose, if you are asked to wear a t-shirt and a pair of pants out of 2 t-shirts and 3 pairs of pants?



Example

1 How many ways are there to choose a male student out of three students (Ashraf - Mohamed - Hassan) and a female student out of two students (Samar - Mona)?

Solution

In this example, we find that it is easy to know the number of ways of choosing. For example, we can choose Ashraf, Samar or Ashraf, Mona or Mohamed, Mona or Hassan, Samar .. etc. We are going to express that by the following graphical diagram called tree diagram:

Male students	Female students	Choice
Ashraf	Samar	Ashraf، Samar
	Mona	Ashraf، Mona
Mohamed	Samar	Mohamed، Samar
	Mona	Mohamed، Mona
Hassan	Samar	Hassan، Samar
	Mona	Hassan، Mona

The number of ways to choose a male student out of three students = 3 ways.

The number of ways to choose a female student out of two students = 2 ways

$\therefore$ Number of ways of choosing = $3 \times 2 = 6$ ways

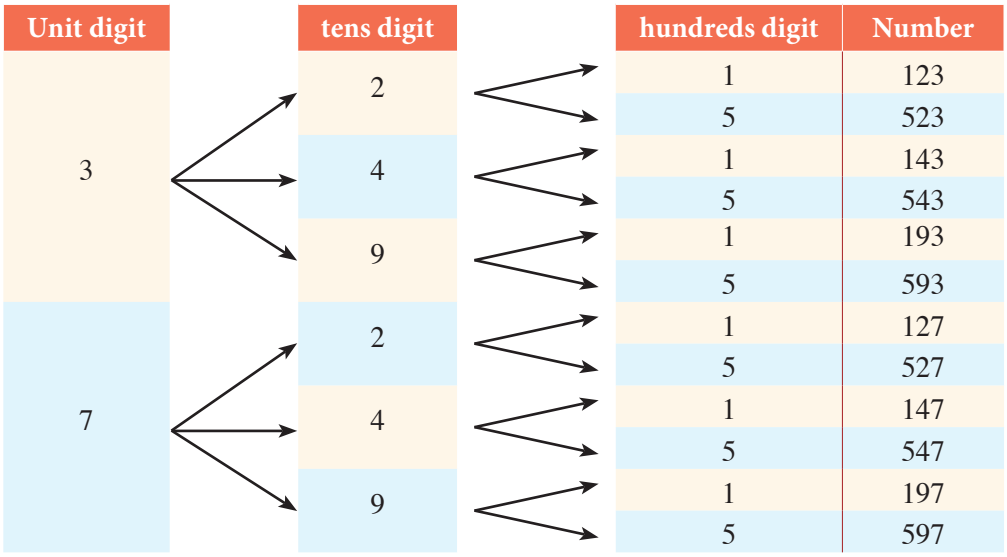
Try to solve

1 In **think and discuss** , How many possible ways of choosing are there?

Example

2 How many three- digit numbers can be formed so that the unit digit is from the elements of {3, 7} the tens digit is from the elements of {2, 4, 9} and the hundreds digit is from the elements of {1, 5} ?

Solution



Unit digit	tens digit	hundreds digit	Number
3	2	1	123
		5	523
	4	1	143
		5	543
	9	1	193
		5	593
7	2	1	127
		5	527
	4	1	147
		5	547
	9	1	197
		5	597

From the tree diagram, we find that:

The number of ways to choose the unit digit × number of ways to choose tens digit × number of ways to choose hundreds digit $= 2 \times 3 \times 2 = 12$ ways

The previous examples support the following definition:

Learn

Fundamental Counting Principle

Definition: If the number of ways to perform a certain task is m_1 way , the number of ways to perform another certain task is m_2 way and the number of ways to perform a third certain task is m_3 way and so on ..., then the number of ways to perform these tasks together $= m_1 \times m_2 \times m_3 \times ... \times m_n$

Example

(3) How many ways can khaled have a meal out of three meals (Liver - chicken - fish) and a beverage out of three beve rages (orange - lemon - Mango)?

Solution

Number of ways to choose a meal $= 3$ ways and the number of ways to choose a drink $= 3$ ways.

The total number of ways to choose $= 3 \times 3 = 9$ ways.

Try to solve

(2) A restaurant presents 6 types of pies, 4 types of salads and 3 types of beverages. How many meals can the restaurant present daily so that a meal includes a type from each of pies, salads and beverages?

Example **Conditional Counting Principle**

(4) How many different three -digit numbers can be formed from the numbers {0 , 1 , 2 , 3 , 4}?

Solution

Start with the conditional hundreds digit (Zero can't be used left side)

place	ones	tens	hundreds
No of ways	3	4	4

Number of ways of choosing the digit in hundreds digit = 4

Number of ways of choosing the digit in tens digit = 4

Number of ways of choosing the digit in unit digit = 3

∴ The total number of ways of choosing = $4 \times 4 \times 3 = 48$ ways

Try to solve

(3) How many ways can a different four -digit number be formed from the numbers {2, 3, 4, 7}, so that the tens digit is even.

Exercises (2 - 1)

Choose the correct answer:

(1) The number of ways of sitting 4 students on 4 seats in a row equals :

(a) 1 (b) $4 + 4$ (c) 4×4 (d) $4 \times 3 \times 2 \times 1$

(2) The number of the different two - digit numbers taken off from the numbers { 5, 3, 0, 2} equals:

(a) 3×2 (b) 4×2 (c) 3×3 (d) 3×4

(3) The number of the different odd three-digit numbers taken off from the numbers {2 , 3, 6, 8} equals:

(a) $8 \times 6 \times 3$ (b) $4 \times 3 \times 3$ (c) $4 \times 3 \times 2$ (d) $2 \times 3 \times 1$

(4) How many three -digit numbers can be formed from the elements {2, 3, 5}?

(5) How many different four - digit numbers can be formed taken off from the elements {2, 3, 6, 8} so that the unit digit is 6?

(6) The licence plates of cars in a governorate start with three letters followed by three digits except Zero. How many plates can be got assuming that there is no repetition for any letter or digit in the licence plates?

(7) How many different three-digit numbers can be taken off from the numbers {2, 5, 8, 9} so that these numbers are less than 900?

(8) If you know that the set of the numbers of mobile networks in a country is made up of an eleven-digit number. If the number (025) is constant on left side, find the greatest number of phone lines which the mobile network can stand .

Factorial of a Number - Permutations

2 - 2

We will learn
- Factorial of a number
- Permutations

Key - term
- Factorial of a Number
- Permutatuins
- Sub-Permutation

Matrials
- Scientific calculator
- Graphical programs

Think and discuss

Use what you learned in the previous lesson to answer the following questions:

1) How many ways can 4 students sit on three seats in a row?

2) How many ways can 5 racers stand on the edge of a swimming pool to jump?

Learn

Definition 1

Factorial: The factorial of a positive integer n is written as $\lfloor n$ and equals the product of all the positive integers which are lesser than or equal n where:

$$\lfloor n = n\,(n-1)\,(n-2)\ldots 3 \times 2 \times 1$$

Notice that

- When n = 0 then $\lfloor 0 = 1$
- When n = 1 then $\lfloor 1 = 1$
- $\lfloor 4 = 4 \times 3 \times 2 \times 1 = 4 \lfloor 3$,
 $\lfloor 6 = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6 \lfloor 5$

In general: $\lfloor n = n \lfloor n-1$ where $n \in Z^+$

Example

1 **a** Find $\frac{\lfloor 10}{\lfloor 8}$ **b** If $\lfloor n = 120$ find the value of n

Solution

a $\frac{\lfloor 10}{\lfloor 8} = \frac{10 \times 9 \lfloor 8}{\lfloor 8} = 10 \times 9 = 90$

b $\lfloor n = 5 \times 4 \times 3 \times 2 \times 1$ $\quad \therefore \lfloor n = \lfloor 5$ then n = 5

Try to solve

1 **Find:** **a** $\frac{\lfloor 15}{\lfloor 12}$ **b** $\frac{\lfloor 7}{\lfloor 5} + \frac{\lfloor 9}{\lfloor 7}$

Example

2 Find the solution set of the equation:- $\frac{\lfloor n}{\lfloor n-2} = 30$

Solution

$\because \frac{\lfloor n}{\lfloor n-2} = \frac{n(n-1)\ \lfloor n-2}{\lfloor n-2} = 30$ $\quad \therefore\ n(n-1) = 6 \times 5$ $\quad \therefore\ n = 6$

Try to solve

(2) If $\frac{1}{\lfloor n} + \frac{2}{\lfloor n+1} = \frac{56}{\lfloor n+2}$ find the value of n

Critical thinking: if: $\lfloor a = \lfloor 0$ find the value of a.

Permutations

Introductory example: How many ways can different three-digit numbers be formed from the set of numbers {2, 3, 5} ?

The numbers are: 532 , 352 , 523 , 253 , 325, 235. Each number of these numbers is called a permutation of the numbers

and its number $= 3 \times 2 \times 1$ **and is written as** ${}^{3}p_{3}$ **and is read as** (3 p 3).

The following table illustrates that:

Unit digit	Tens	Hundreds
1	2	3

So, the permutation for a number of objects is to put them in a certain arrangement

Definition 1

The number of permutations of n different objects taking r at a time is denoted by the symbol ${}^{n}p_{r}$ where:

${}^{n}p_{r} = n(n-1)(n-2)\ldots(n-r+1)$ where $r \leqslant n$, $r \in N$, $n \in Z^{+}$

${}^{n}p_{0} = 1$

For example:

- ${}^{6}p_{3} = 6 \times 5 \times 4 \times \frac{3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{\lfloor 6}{\lfloor 6-3}$
- ${}^{7}p_{5} = 7 \times 6 \times 5 \times 4 \times 3 \times \frac{2 \times 1}{2 \times 1} = \frac{\lfloor 7}{\lfloor 7-5}$

${}^{6}p_{3} = \frac{\lfloor 6}{\lfloor 6-3}$

of the previous , we deduce that:

${}^{n}p_{r} = \frac{\lfloor n}{\lfloor n-r}$ where $r \in N$, $n \in Z^{+}$, $r \leqslant n$

Example

(3) Find the value for reach of the following:

a ${}^{7}p_{4}$ **b** ${}^{4}p_{4}$ **c** ${}^{4}p_{3}$

Solution

a ${}^{7}p_{4} = 7 \times 6 \times 5 \times 4 = 840$ **b** ${}^{4}p_{4} = 4 \times 3 \times 2 \times 1 = 24$

c ${}^{4}p_{3} = 4 \times 3 \times 2 = 24$. What do you notice in the two phrases b and c?

using calculator

Permutations are denoted by the symbol ${}^{n}p_{r}$ in the calculator and we use the buttons Shift × To calculate the value of ${}^{5}p_{2}$ by the calculator, press the following buttons consecutively:

5 Shift × 2 = The answer = 20

Try to solve

3 Calculate the value of the following: a $^5p_2 + {}^6p_3$ b $\frac{^5p_5}{^5p_4}$

Example

4 Find the number of the different ways, for 5 students to sit on 7 seats in one row.

Solution

We have 7 seats. Among the 7 seats, 5 should be selected each time

$\therefore$ Number of ways $= {}^7p_5 = 7 \times 6 \times 5 \times 4 \times 3 = 2520$

Use the calculator: 7 SHIFT × (npr) 5 =

Try to solve

4 How many words can be formed from five different Alphabetic letters?

Example **(Arranging (ordering) in a circle)**

5 How many ways can 4 persons sit on 4 seats in the form of a circle ?

Solution

In this case , the first person sits in one way, second person sits in three ways, third person sits in 2 ways and the last person sits in an only way.

The number of ways of their arrangement $= 3 \times 2 \times 1 = \lfloor 3 = 6$

Critical thinking: Can you deduce how many ways n of persons can be arranged in a circle?

Try to solve

5 How many ways can 9 persons in 9 seats be arranged in the form of a circle?

Example

6 If $^7p_r = 840$, find the value of $\lfloor r - 4$

840	7
120	6
20	5
4	4
1	

Solution

Start by dividing the number 840 by 7, then divide the quotient by 6, then divide the resulted quotient by 5 and so on till you reach number 1

$\therefore$ Number $840 = 7 \times 6 \times 5 \times 4 = {}^7p_4$

$\therefore\ {}^7p_r = {}^7p_4$ $\therefore\ r = 4$ $\therefore\ \lfloor r - 4 = \lfloor 0 = 1$

Try to solve

6 If $^9p_{r-1} = 504$, find the value of $\lfloor r + 1$

Critical thinking: 1) Find the value of: $^7p_7$, $\lfloor 7$. **What do you notice?**

Exercises (2 - 2)

Choose the correct answer:

1. How many ways can a president and vice president be selected from a 12 - member committee?
 a 2 **b** 23 **c** 66 **d** 132

2. If ${}^{5}p_{r} = 60$, then r equals:
 a 4 **b** 3 **c** 2 **d** 5

3. If ${}^{n}p_{3} = 120$, then the value of n equals:
 a 6 **b** 5 **c** 4 **d** 3

4. The number of ways to arrange the letters of the word HELP equals:
 a 4 **b** 9 **c** 10 **d** 24

5. The number of ways to select a different two - digit number of the set of numbers {3, 4, 5, 6} equals:
 a 48 **b** 30 **c** 12 **d** 4

6. The number of ways to arrange 7 children in a circle equals :
 a 1 **b** 7 **c** 720 **d** 5040

7. A phone number is made up of 8 places

9	c						

 c should be a number of 3, 4, 5, 8 while the rest of the digits are made up of any unconditional digit. How many different telephone numbers are available?
 a 499999 **b** 4000000 **c** 4999999 **d** 10000000

8. How many ways can Hossam have a meal out of three meals (Kofta - chicken - fish) and a beverage out of two beverages (Juice - soft drink)? Represent it using the tree diagram.

9. How many ways can a two -digit number be formed from the numbers 1 , 2 , 3 , 4?

10. How many ways can a different two -digit number be formed from the numbers 1 , 2 , 3 , 4?

11. How many ways can a different two -digit even number be formed from the numbers 1 , 2 , 3, 4?

12. How many ways can a committee of a man and a woman be formed from 3 men and 4 woman?

13 An Ice-cream shop offers three sizes (small - medium - large) and five flavors (strawberry , mango, lemon , milk, chocolate)
How many ways are available to buy an ice -cream cone?



14 From the set of the letters {a , b, c , d, e , f}, find
- **a** Number of ways to select a letter.
- **b** Number of ways to select two different letters.

15 **Find the value of the following:**
- **a** $\lfloor 7 \div \lfloor 5$
- **b** $3 \lfloor 2 - \lfloor 3$
- **c** ${}^{5}p_{3} \times \lfloor 2$
- **d** ${}^{3}p_{3} \times {}^{2}p_{2}$
- **e** ${}^{8}p_{1} + {}^{8}p_{2}$
- **f** ${}^{7}p_{0} + {}^{7}p_{7}$

16 **Find the value of n which satisfies the following:**
- **a** $\lfloor n = 24$
- **b** $\frac{\lfloor n+1}{\lfloor n-1} = 42$
- **c** ${}^{15}p_{n} = 2730$
- **d** ${}^{n}p_{0} + {}^{n}p_{1} + {}^{n}p_{2} = 50$

17 Find the value of n if:
- **a** ${}^{7}p_{n} = 210$
- **b** $n \lfloor 2n - 1 = 12$

18 If ${}^{n}p_{4} = 14 \times {}^{n-2}p_{3}$ find the value of n.

19 Find the number of ways to select a president, vice president and a secretary out of a ten-person committee.

20 How many ways can the physical education instructor choose three students (one after another) to participate in the teams of soccer , basketball and volleyball respectively from eight students?

21 Prove that: $\frac{\lfloor n+2}{\lfloor n} = {}^{n+2}p_{2}$

Activities

1- If x = {2 , 3 , 5 , 7 , 9}
- **a** How many ways can a different two-digit number be formed from these numbers?
- **b** How many ways can a two-digit number be formed from these numbers?
- **c** How many ways can a different three-digit number be formed from these numbers?
- **d** How many ways can a different three-digit number less than 500 be formed from these numbers?
- **e** How many ways can a different four-digit number whose unit digit is 2 be formed from these numbers?

Unit Two

2 - 3 Combinations

We will learn

- The concept of combinations and simple applications on them.
- Pascal's triangle.

Key - term

- Combinations
- Elements
- Order
- Committee
- Subset

Matrials

- Scientific calculator
- Computer

Tip

Combination can be written in the form of ${}^{n}C_{r} = \binom{n}{r}$

Introduction



Two clubs of a four - club set { a , b, c , d} are to be selected for a soccer match, then the possible permutations are:
(a, b), (a, c), (a, d), (b, a) , (b, c), (b, d), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c).
From the previous data, we notice that selecting (a, b) is different from selecting (b, a) and so on...
If we want to select from the previous disregarding the order, then all the possible choices are: {a, b}, {a, c}, {a, c}, {b, c}, {b, d}, {c, d}. and each choice of these choices is called" Combination"

Combinations

Definition

The number of combinations formed from r of objects chosen from n elements at the same time is ${}^{n}C_{r}$ where, $r \leqslant n$, $r \in N$, $n \in Z^{+}$

In the previous introduction , we find that:
the number of combinations of two elements taken from four elements is denoted by the symbol ${}^{4}C_{2}$ and is read as (4 c 2) or by the symbol $\binom{4}{2}$ and is read as 4 c 2. In the introduction above, we notice that the number of ways of choosing = 6 ways

i.e.: ${}^{4}C_{2} = \frac{{}^{4}P_{2}}{\lfloor 2} = \frac{4 \times 3}{2 \times 1} = 6$,

$${}^{n}C_{r} = \frac{{}^{n}P_{r}}{\lfloor r}$$

Example

1 Find the value for each of the following

a ${}^{7}C_{5}$ b ${}^{7}C_{2}$ (what do you notice)?

Solution

a ${}^{7}C_{5} = \frac{{}^{7}P_{5}}{\lfloor 5} = \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2 \times 1} = 21$ b ${}^{7}C_{2} = \frac{7 \times 6}{2 \times 1} = 21$

We notice that: ${}^{7}C_{5} = {}^{7}C_{2}$ $(5 + 2 = 7)$

Important corollaries: ${}^{n}c_{r} = \frac{\lfloor n}{\lfloor r \lfloor n - r}$ ${}^{n}C_{r} = {}^{n}C_{n-r}$

Try to solve

(1) Find the value of ${}^{12}C_{9}$: ${}^{17}C_{14}$ without using the calculator.

Activity **Using the calculator**

The buttons SHIFT ÷ can be used from left to right to write the symbol of combinations(${}^{n}C_{r}$)

1) Use the calculator to find the value of ${}^{5}C_{4} + {}^{7}C_{2}$

Solution

Press the following buttons consecutively

start ⟶ 5 SHIFT ÷ 4 + 7 SHIFT ÷ 2 =

The result = 26

Example

(2) If : ${}^{28}C_{r} = {}^{28}C_{2r - 47}$, find the values of r.

Solution

$\because$ ${}^{28}C_{r} = {}^{28}C_{2r - 47}$

either : $r = 2r - 47$ i.e.: $r = 47$

it is greater than the value of n, so it is refused.

or : $r + 2r - 47 = 28$ $\therefore 3r = 75$ $\therefore r = 25$

Try to solve

(2) If ${}^{28}C_{r} = {}^{28}C_{2r - 5}$, then find the value of r.

Example

(3) How many ways can a four - person team be chosen from a 9 - person set?

Solution

So that the choice disregards the order, then each choice is called a combination.

Number of choices = ${}^{9}C_{4} = \frac{{}^{9}P_{4}}{\lfloor 4} = 126$

Try to solve

(3) 7 people have participated in a chess game so that a game is held between each two players. How many matches are there?

Example **Counting principle**

4 How many ways can a committee of two men and a woman be selected out of 7 men and 5 women?

Solution

Number of ways to select 2 men out of 7 men = ${}^{7}C_{2} = 21$

Number of ways to select a women out of 5 women = ${}^{5}C_{1} = 5$

According to the counting principle, the number of ways to form the committee = $21 \times 5 = 105$ ways

Critical thinking: How many ways can a committee of 4 men and 3 women be selected out of 6 men and 5 women ?

Try to solve

4 How many ways can a five-member committee formed from 3 male students and 2 female students be selected out of a class contains 10 male students and 8 female students?

Activity **Pascal's triangle**



Blaise pascal (1623 - 1662):

Blaise pascal is a french philosopher, mathematician and physicist. He had stated the theory of probabilities and designed a triangular array of numbers called pascal's triangle in calculating the probabilities. Furthermore, he had invented a calculator to perform the addition and multiplication operations

Check the opposite number triangle , then answer the following questions:

1- What do you notice about how numbers are written in this triangle?

2- Is there a relation among the number of elements of each row and the row directly next to it?

3- Is there a symmetry among the numbers existed on the two sides of the triangle?

			1											
		1		1										
	1		2		1									
1		3		3		1								
	1		4		6		4		1					
1		5		10		10		5		1				
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

After performing the activity, we can notice that:

- **First row:** represents (n = 1) of the elements taken off from r = 0 or r =1

 So: ${}^{1}C_{0} = 1$, ${}^{1}C_{1} = 1$

- **Second row:** represents (n = 2) of the elements taken off from r = 0 or r =1 or r = 2 in each time.

 then: ${}^{2}C_{0} = 1$, ${}^{2}C_{1} = 2$, ${}^{2}C_{2} = 1$ and so on.

As we notice that:

- Each row starts with one because ${}^{n}C_{0} = 1$, and ends in one because ${}^{n}C_{n} = 1$
- Each number in any row except for the first row equals the sum of the two numbers located above it in the row directly above it.
- **In the third row , we find:** 1 , 1 + 2 , 2 + 1, 1
- **In the fourth row , we find:** 1 , 1 + 3 , 3 + 3 , 3 + 1 , 1 and so on.
- There is a symmetry about the number located at the middle of the row (if n is even)
- There is a symmetry about the two numbers located at the middle of the row (if n is odd)
- This coincides the previous relation ${}^{n}C_{r} = {}^{n}C_{n-r}$

Application on the activity:

Prove that: ${}^{5}C_{0} + {}^{5}C_{1} + {}^{5}C_{2} + {}^{5}C_{3} + {}^{5}C_{4} + {}^{5}C_{5} = 2^{5}$

Exercises (2 - 3)

Choose the correct answer from those given:

1. The number of ways to choose 3 people out of five people equals.....
 a) 15 b) 10 c) 20 d) 35

2. The number of ways to answer 4 questions only out of a 6 - question exam equals.....
 a) 30 b) 15 c) 24 d) 10

3. The number of ways to choose a red ball and a white ball out of 5 red balls and 3 white balls equals.....
 a) 15 b) 8 c) 60 d) 2

Answer the following questions:

4. Calculate the value of ${}^{6}C_{3}$, ${}^{9}C_{1}$, ${}^{12}C_{11}$ and ${}^{100}C_{0}$

5. If ${}^{n}C_{3} = 120$, find the value of ${}^{n}C_{n-9}$

6. If ${}^{n+1}C_{4} = \frac{5}{2}\ {}^{n}C_{3}$, find the value of n.

7. If ${}^{n}C_{3} = 30\frac{1}{3}\,n$, find the value of n.

8. How many ways can a five-people committee take the majority of a decision?

9 How many ways can a five - student activity committee formed from three male students and two female students out of a class contains 10 male students and 8 female students?

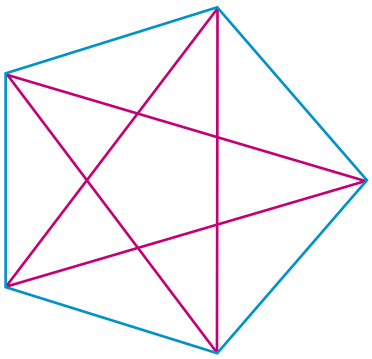
10 Write in terms of permutations each of:

a $^{8}C_{3}$ **b** $^{19}C_{2}$ **c** $^{5}C_{0}$ **d** $^{x}C_{x-y}$

11 Use the form $^{n}C_{r}$ to write each of:

a $\frac{^{8}p_{2}}{\lfloor 2}$ **b** $\frac{^{9}p_{3}}{\lfloor 3}$ **c** $\frac{^{10}p_{4}}{\lfloor 4}$ **d** $\frac{^{8}p_{0}}{\lfloor 0}$

Activities



You have previously learned that the diagonal of a geometric figure is the line segment connecting two non-consecutive vertices and you also knew that:

The number of the triangle diagonals = zero

The number of the quadrilateral diagonals = 2

The number of the pentagon diagonals = 5

Can you find the number of the diagonals of the following figures:

1) Hexagon, octagon and decagon.

2) Can you use combinations to write the rule by which you can find the diagonals of any geometric figure? Check with your instructor using internet.

Unit summary

Fundamental counting principle: If the number of ways to perform a certain task equals n way an the number of ways to perform another task equals m way, then the number of ways to perform the first and second tasks together = $m \times n$ ways.

Permutations: If X is a set of n - elements, then each order for some or all the elements of this set is called a permutation.

Let X be a set of n-elements, then each order of r-element can be formed from the set X is called a permutation where $0 \leqslant r \leqslant n$ and the number of the sub-permutations formed from r element taken off from a set of n elements is denoted by $^{n}p_{r}$ (and read as n p r)

Theory: The number of the possible permutations from a set of n - elements equals $n(n-1)(n-2) \ldots \times 3 \times 2 \times 1$ the previous product is denoted by the symbol $\lfloor n$ and read as (factorial of n)

Combinations: If X is a set of n- elements , then each choice of r- elements where $(0 \leqslant r \leqslant n)$ can be taken off from the elements of the set X disregarding the order is called a combination.

- The number of possible combinations formed from r element and taken off from n element is denoted by the symbol $^{n}C_{r}$ (and read as n C r)

General exercises

Complete:

1. The number of ways to form a different two -digit number of the set of the numbers 1, 2, 3, 4 equal.....

2. If you are asked to make a three-digit code doesnot include zero for an iron treasury, then the number of ways equals.....

3. If ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^n$, then n =.....

4. If $\lfloor n = \lfloor 1$ then n = ...

Choose the correct answer:

5. A man has wanted to buy a car of three models {a, b, c} and he wanted to select one of colors:{white - red - silver - black}. How many ways can this man choose a car?

 a) 7 b) 12 c) 14 d) 24

6. How many ways can a different three- digit number be formed from the numbers 1, 3, 6, 7

 a) 9 b) c) 24 d) 64

7. if X = {x:x ∈N and $1 \leqslant x \leqslant 5$} and Y = {(a, b): a ,b∈ x and a ≠b}. How many elements are there in Y?

 a) 7 b) 10 c) 20 d) 25

8. ${}^{12}C_4 + {}^{12}C_3$ equals

 a) 715 b) 710 c) 716 d) 720

9. If ${}^nP_r = 336$, ${}^nC_r = 56$ then n , r are

 a) (3, 2) b) (8 , 3) c) (7, 4) d) (7, 3)

10. If ${}^nC_{10} = {}^nC_{14}$, then ${}^{25}C_n$ equals.

 a) 24 b) 25 c) 1 d) 49

11. Calculate the value of each:

 a) ${}^5C_5$ b) $\lfloor 6$ c) ${}^{23}P_1$

 d) ${}^{50}C_{49}$ e) $\frac{\lfloor 13}{\lfloor 14}$ f) ${}^4P_4$

12 What is the value of $\frac{\lfloor n-5}{}$, if ${}^{2n-1}C_3 = 84$?

13 What is the value of ${}^{2r}C_4$, if ${}^{8}p_{r+1} = 336$?

14 If ${}^{n-m}p_3 = 210$, ${}^{n+m}C_4 = 715$, find the value of m and n.

15 How many ways can 7 students be selected from 10 students to take a historical trip?

16 If 3 students are selected from (n) number of students to attend a seminar so that the number of ways of choosing is 10 ways, find the number of students.

17 How many ways can a committee of a man and two women be selected from 7 men and 5 women?

18 It is wanted to choose a teacher out of 4 teachers to train the Olympics students in mathematics and another teacher to prepare the test, find the number of ways of choosing.

19 The number of the premier league soccer teams is 12 teams. If each team plays two matches with each team of other teams , find the total number of the premier league matches .

20 Find the solution set of the following equations:

a $\frac{\lfloor n+3}{\lfloor n+1} = 42$ b $12 \times \lfloor n = \lfloor n+2$ c $\lfloor n-4 = 120$

d $\lfloor n-5 = 1$ e ${}^{n}C_3 = 84$ f ${}^{12}C_r = {}^{12}C_{2r+3}$

Accumulative test

Choose the correct answer:

1 The number of ways to form a different three -digit prime number of the set of the numbers 3, 4, 5 is.....

a 6 b 3 c 1 d zero

2 The number of ways to form the number 5476 from the numbers 4, 5, 6, 7 is

a 24 b 16 c 1 d zero

3 The number of ways to form a 3-digit number of 5 numbers except zero is

a $5 \times 4 \times 3$ b $5 + 4 + 3$ c $5 \times 5 \times 5$ d $3 \times 2 \times 1$

4 If the letters of the word CAT are used to write a different three- letter word, then number of the words resulted equals:

a 3 b 6 c 9 d 1

5 ${}^{9}P_6$ equals

a $9 \times 8 \times 7$ b $9 \times 8 \times 7 \times 6 \times 5 \times 4$ c $\lfloor 9$ d $\lfloor 6$

6 Find:

a $^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

b $^5C_0 - {}^5C_1 + {}^5C_2 - {}^5C_3 + {}^5C_4 - {}^5C_5$

7 Find: $^8p_3 + {}^8p_2 + {}^8p_1$

8 5 points are on a plane so that there are not any three points on a straight line. How many triangles can be formed from these points?

9 How many different three -digit numbers can be formed from the set of numbers {1, 2, 3, 4, 5, 6}?

10 Find the value of each:

a $\frac{\lfloor 99}{\lfloor 100}$

b $\lfloor 0 + \lfloor 1 + \lfloor 2 + \lfloor 3$

c $\frac{\lfloor 19}{\lfloor 20}$

d $\lfloor 5 - \lfloor 4$

e $\frac{\lfloor 4}{\lfloor 0}$

11 If $x = \{-2, -1, 0, 1, 2, 3\}$
and $z = \{\{a, b\}: a \neq b, a, b \in x\}$, find the number of z elements.

12 Find the value of n in each of the following cases:

a $\lfloor n = 720$

b $n \lfloor 2n - 1 = 60$

c $^nC_{n-2} = 28$

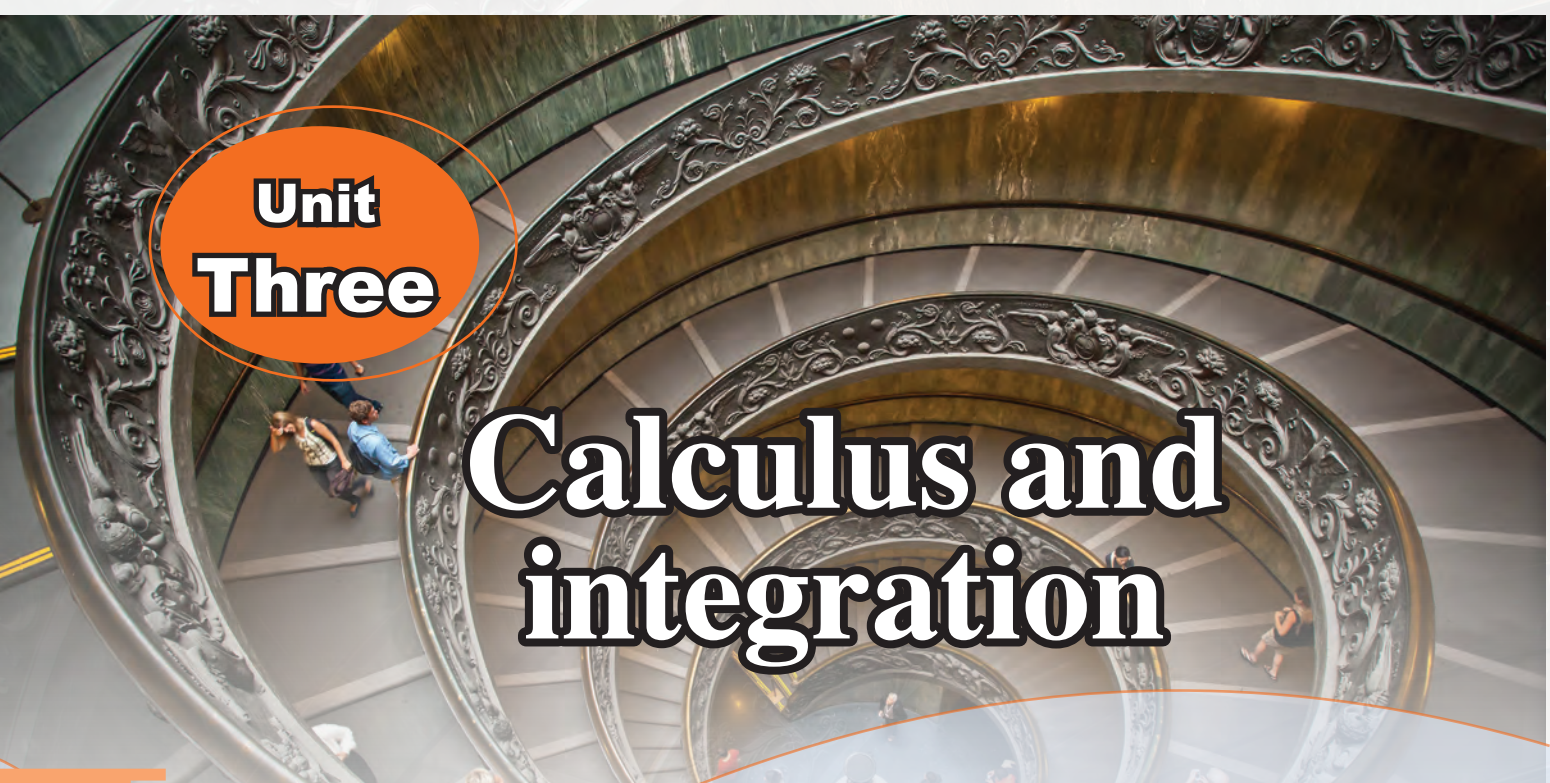
d $^{n+2}p_4 = 56 \times {}^n p_2$

e $\frac{1}{\lfloor n} = \frac{2}{\lfloor n+1}$

f $^m p_n = 24\ {}^m C_n$

If you can not answer any of these questions , you can use the following table:

Question no	1	2	3	4	5	6	7	8	9	10	11	12
Back to	counting principle	counting principle	counting principle	counting principle	permutations	combinations	permutations	combinations	counting principle	factorial of a number	combinations	permutations combinations



Unit Three

Calculus and integration

Unit introduction

There is no doubt aware that the calculus one of the modern science, which are used in many branches of knowledge, and in various scientific contexts such as engineering Physics, medicine, economics and geography, calculus is interested in studying the differentiation changes and differences)variable quantities(through accounts include averages, observes and rates of change compared to the rate of change is attributed to the change in average temperatures - price - speed - production ... as well as the study of the behavior Functions such as the functions of costs and income in the economy; to maximize profit or study the effect of taking a particular drug in reducing the duration of treatment or study schemes belt Earthquakes when planning for the construction, urban stress and expense rates, etc., The calculus is looking to find a quantity Bmalomah rate Change, and is used to calculate the area under the curve, and the amount of labor resulting from the impact of a changing force, and includes calculus operations With infinitely small quantities in contrast to algebra. Finally, we sought to understand the minutes of this article will help solve a lot of problems with overlapping many of the minute, which are built by Renaissance Science Scientific and cultural that we wish for our country..

Unit objectives

By the end of this unit, the student should be able to:

- Recognize the concept of variation function, average rate of change function and the rate of change function.
- Conclude the first derivation function.
- Recognize the geometrical meaning of the first derivative (the slope of the tangent).
- Identify some derivative rules .
 - the derivative of the constant function.
 - the derivative of the function $f : f(x) = x^n$
 - the derivative of the function $f : f(x) = x$
 - the derivative of the function $f : f(x) = a\,x^n$
 - the derivative of the sum and difference between two function
 - the derivative of product of two functions.
 - the derivative of the quotient of functions.
 - the derivative of the composite function - the chain rule.
 - the derivative of the function $y = (f(x))^n$
- Use the derivatives in geometrical applications like finding the equations of the tangent to a curve at a point on it.
- Recognize the concept of integral -antiderivative.
- Recognize the following integral rules:
 - $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \, , n \neq -1$
 - $\int a f(x) \, dx = a \int f(x) dx$, where a is a constant
 - $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
 - $\int (a x + b)^n \, dx = \frac{1}{a(n+1)} (a x + b)^{n+1} + C, n \neq -1$

Key terms

- Variation
- Average Rate of Change
- Rate of Change
- Differentiation
- Differintiable Function
- First Derivative
- Product
- Quotient
- Antiderivative
- Integration

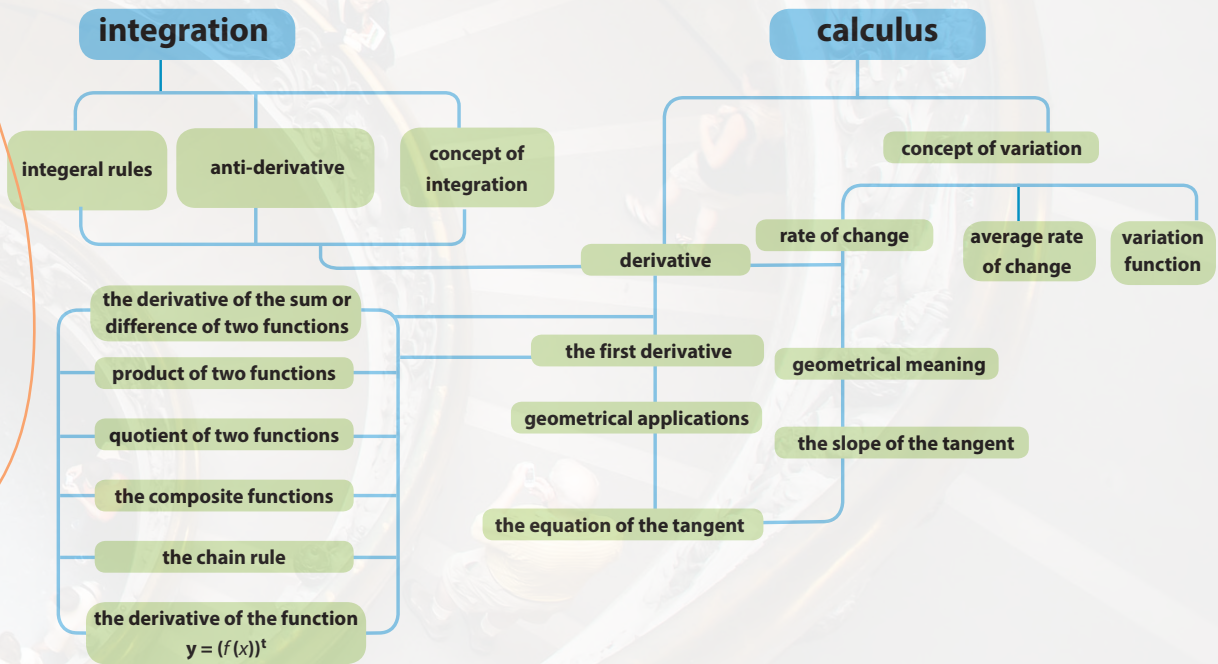
Unit Lessons

Lesson (3 - 1): The rate of change.
Lesson (3 - 2): Differentiation.
Lesson (3 - 3): Derivative rules.
Lesson (3 - 4): Integration

Materials

Scientific calculator - Computer - Graphing program

Unit planning guide



Unit Three

3-1 Rate of Change

We will learn

- The concept of the variation function.
- The concept of the average rate of change.
- The concept of the rate of change.

Key - term

- Function of Variation
- Average rate of Change
- Rate of Change

matrials

- Scientific calculator

Think and discuss

Why leave spaces between the railway bridges or metal rods breaks? If the temperature increases from 30° to 42° in a time period or decreases from 48° to 22° in another time period Calculate the change in temperature for each period ... What do you notice?

If the length of one of the rails is L meters and changed its temperature x from x_1 to x_2 then we say that a change has occurred in the temperature and be: The change in the value of x = the value of x at the end of the change - the value of x at the beginning of change

Is the length of the rails varies depending on the temperature change?

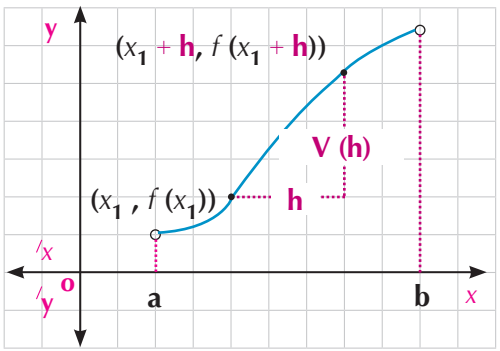
Learn **The Variation Function**

If $f :]a, b[\longrightarrow \mathbb{R}$ where $y = f(x)$ then any change in the value of x from x_1 to x_2 in the domain of f corresponds change in the value of y from $f(x_1)$ to $f(x_2)$ then :

The value of the change of $x = \Delta x$ (read as delta x) $= x_2 - x_1$,

The value of the change of $y = \Delta y = f(x_2) - f(x_1)$

and by considering $(x_1, f(x_1))$ a point on the curve of the function f , then for each change in the coordinate of x from x_1 to $x_2 = x_1 + h$ such that $x_1 + h \in]a, b[$, $h \neq 0$ then there is a corresponding change in the y-coordinate given by the relation :



$V(h) = f(x_1 + h) - f(x_1)$ the function V is called the variation function of the function f at $x = x_1$

Notice:

Each of symbols Δx , h represents the variation of x

Example

1 If $f(x) = 3x^2 + x - 2$ and x changes from 2 to 2 + h find the variation function V, then calculate the variation of f when

a $h = 0.3$ b $h = -0.1$

Solution

$\because f(x) = 3x^2 + x - 2$, x changed from 2 to 2 + h

$\therefore x_1 = 2$, $f(2) = 3 \times 4 + 2 - 2 = 12$, and :

$$f(2 + h) = 3(2 + h)^2 + (2 + h) - 2 = 12 + 12h + 3h^2 + 2 + h - 2$$
$$= 3h^2 + 13h + 12$$

$$V(h) = f(2 + h) - f(2)$$
$$= (3h^2 + 13h + 12) - 12 = 3h^2 + 13h$$

a when $h = 0.3$

$V(0.3) = 3(0.3)^2 + 13 \times 0.3$

$= 4.17$

b when $h = -0.1$

$V(-0.1) = 3(-0.1)^2 + 13(-0.1)$

$= -1.27$

Try to solve

1 If $f(x) = x^2 - x + 1$ find the variation function V when $x = 3$ then calculate:

a V(0.2) b V(- 0.3)

Learn

The Average rate of change function

when we divide the variation function V by h where $h \neq 0$ we get a new function called the function of the average rate of change of f at $x = x_1$ where :

$$A(h) = \frac{V(h)}{h} = \frac{f(x_1 + h) - f(x_1)}{h} \quad \text{or} \quad \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example

2 If $f : [0, \infty[\longrightarrow \mathbb{R}$ where $f(x) = x^2 + 1$ find :

a The function of the average rate of change of f when $x = 2$ then calculate A (0.3)

b The average rate of change of f when x changes from 3 to 4

Solution

a $f(x_1) = f(2) = (2)^2 + 1 = 5$, $f(x_1 + h) = f(2 + h)$

$\therefore\ f(2+h) = (2+h)^2 + 1 = h^2 + 4h + 5$

$\because A(h) = \dfrac{f(x_1 + h) - f(x_1)}{h}$

$\therefore A(h) = \dfrac{h^2 + 4h + 5 - 5}{h} = h + 4$ then $A(0.3) = 4.3$

b When x changes from 3 to 4 then $x_1 = 3$, $x_2 = 4$

then $f(3) = 9 + 1 = 10$, $f(4) = 16 + 1 = 17$

The average rate of change $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{17 - 10}{4 - 3} = 7$

Try to solve

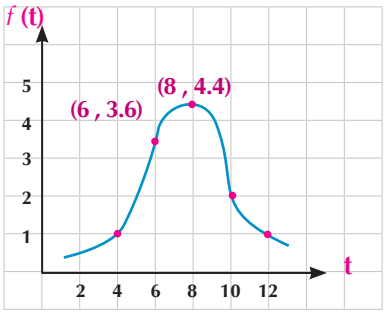
2 If $f(x) = x^2 + 3x - 1$ find :

a The average rate of change function at $x = 2$ then find A (0.2)

b The average rate of change when x changes from 4.5 to 3

Example

3 The opposite graph shows the curve $g = f(t)$ where g is the total sales of one of computer stores estimated in millions of pounds, t the time in months. From the graph find , the average rate of change in the total sales when the time changes from.



a t = 4 to t = 8 **b** t = 8 to t = 10

Solution

a From the graph : $f(8) = 4.4$, $f(4) = 1$

the average rate of change $f = \dfrac{f(8) - f(4)}{8 - 4} = \dfrac{4.4 - 1}{4} = 0.85$ million pound / month

i.e. the average of the total sales increases by 0.85 million pound monthly during this period .

b From the graph : $f(10) = 2$, $f(8) = 4.4$

the average rate of change of $f = \dfrac{f(10) - f(8)}{10 - 8} = \dfrac{2 - 4.4}{2} = -1.2$ million pound / month

i.e. the average of the total sales decreases by 1.2 million pound monthly during this period.

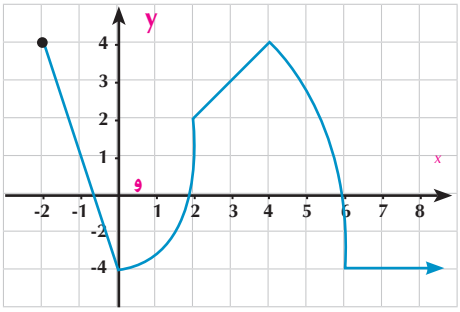
Try to solve

3 Using the graph shown in example 3 . Find the average rate of change in the total sales when the time changes from :

a t = 4 to t = 6 **b** t = 6 to t = 10 **c** t = 4 to t = 12

Critical thinking :

The opposite graph shows the curve of the function f where $y = f(x)$. Determine the intervals over the average rate of change of f is constant. Explain your answer.



Learn

The rate of change function

If $f :]a, b[\longrightarrow \mathbb{R}$ where $y = f(x)$, x_1 , $x_1 + h \in]a, b[$ then:

the function of the rate of change of f at $x_1 = \lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h} = \lim\limits_{h \to 0} A(h)$ on condition that the limit exist.

Example

4 Find the function of the rate of change of f when $x = x_1$ then find this rate at the indicated values of x

a $f(x) = 3x^2 + 2$, $x = 2$ **b** $f(x) = \dfrac{2}{x - 1}$, $x = 3$

Solution

a $\because f(x) = 3x^2 + 2$ $\therefore$ then $x = x_1$ then $f(x_1) = 3x^2_1 + 2$,

$f(x_1 + h)^2 = 3(x_1 + h)^2 + 2 = 3x^2_1 + 6x_1 h + 3h^2 + 2$

the function of the rate of change of $f = \lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{6x_1 h + 3h^2}{h} = \lim\limits_{h \to 0} (6x_1 + 3h) = 6x_1$

when $x = 2$ $\therefore x_1 = 2$ then the rate of change of $f = 6 \times 2 = 12$

b $\because f(x) = \dfrac{2}{x - 1}$ $\therefore$ when $x = x_1$ then :

$f(x_1 + h) - f(x_1) = \dfrac{2}{x_1 + h - 1} - \dfrac{2}{x_1 - 1}$

$= \dfrac{2x_1 - 2 - 2x_1 - 2h + 2}{(x_1 + h - 1)(x_1 + h)} = \dfrac{-2h}{(x_1 + h - 1)(x_1 - 1)}$

the function of the rate of change of $f = \lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h}$

$= \lim\limits_{h \to 0} \dfrac{1}{h} \times \dfrac{-2h}{(x_1 + h - 1)(x_1 - 1)} = \dfrac{-2}{(x_1 - 1)^2}$

when $x = 3$ $\therefore x_1 = 3$ then the rate of change of $f = \dfrac{-2}{(3 - 1)^2} = -\dfrac{1}{2}$

Try to solve

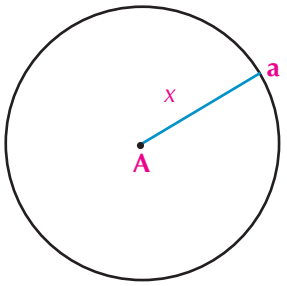
4 Find the function of the average rate of f where $f(x) = \dfrac{3}{x - 2}$ when x changes from x_1 to $x_1 + h$ then deduce the rate of change of f when $x = 5$

Example

5 **Life appliaction:** A stone fall in a still water and a circular wave is formed and increases regularly preserves its circular shape. Find the rate of change of its area with respect to its radius length when the radius length is 3.5cm ($\pi = \frac{22}{7}$)

Solution

Modeling the problem:



Let the radius length of the wave = x cm

$\therefore$ the area of the wave A = $\pi\ x^2$ cm^2

$\therefore$ A = $f(x) = \pi\ x^2$

when x changes from x_1 to $x_1 + h$

then the function of the rate of change of A = $\lim\limits_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$

$$= \lim_{h \to 0} \frac{\pi (x_1 + h)^2 - \pi x_1^2}{h}$$

$$= \pi \lim_{h \to 0} \frac{(x_1 + h)^2 - x_1^2}{h} = 2\pi\ x_1$$

when $x = x_1 = 3.5$ $\therefore$ the rate of change of A = $2 \times \frac{22}{7} \times 3.5 = 22$

Try to solve

5 A squared lamina expands regularly preserves its shape, calculate the average rate of change of the area of the lamina when its side length changes from 3cm to 3.4cm, then calculate the rate of change of its area when its side length is 5cm.

Exercises 3 - 1

Choose the correct answer

1 If the average rate of change of $f = 2.4$ when x changes from 3 to 3.2 then the variation of f equals

a 0.32 b 0.48 c 3.6 d 7.2

2 If the average rate of change of $f = 5$ when x changes from 2 to 4 , $f(2) = 6$ then $f(4)$ equals.....

a - 4 b 7 c 8 d 16

3 The average rate of change of the volume of a cube when its edge length changes from 5cm to 7cm equals

a 125 b 343 c 218 d 109

4 The average rate of change of the function f where $f(x) = x^2 + 3x + 5$ when x changes from 1 to 3 equals.....

a 1 b 3 c 7 d 9

Answer the following:

5 If $f(x) = x^2 + 2x - 1$ find the variation of f when

a x changes from 2 to 2.1

b $x = -2$, $h = 1$

6 Find the function of the rate of change of f when $x = x_1$ then deduce the rate of change of f at the indicated values of x_1 in each of the following :

a $f(x) = 2x^3$, $x_1 = 2$

b $f(x) = \frac{x+1}{x-1}$, $x_1 = 0$

7 **Areas:** Asquared lamina shrinking by cooling preserves its squared shape , calculate the rate of change of the area of the lamina with respect to its side length when the side length is 8cm.

8 **Volumes:** A metal sphere expands by heat preserves its spherical shape , find the rate of change of the volume of the sphere with respect its radius length when the radius length is 7cm.

9 **Find the error :** A rectangular lamina of length double its width , expands regularly by heat and preserves its shape and the same ratio between its dimensions , calculate the rate of change of the area of the lamina with respect to its width when the width is 10cm.

The first solution

let the width x, length $2x$

the area = length × width $\therefore f(x) = 2x^2$

the rate of change $= \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$

$= \lim_{h \to 0} \frac{2h(2x_1 + h)}{h}$

$= \lim_{h \to 0} (4x_1 + 2h) = 4x_1$

when x_1 = 10cm

the rate of change = 40

The second solution

let the length x, width $\frac{1}{2}x$

the area = length × width $\therefore f(x) = \frac{1}{2}x^2$

the rate of change $= \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$

$= \lim_{h \to 0} \frac{h(x_1 + \frac{1}{2}h)}{h}$

$= \lim_{h \to 0} (x_1 + \frac{1}{2}h) = x_1$

when x_1 = 10cm

the rate of change = 10

Agriculture:

10 If the quantity y (measured in Kg) produced by a medium production orange tree, which depends on the number of kilograms x of insecticide used of spray the tree according the relation $y = 100 - \frac{42}{x+1}$ calculate the average rate of change of y when x changes from 1 to 2 .

Unit Three

3 - 2 Differentiation

We will learn

- The first derivative function
- The geometrical meaning of the first derivative (the slope of the tengant)

Key - term

- First Derivative
- Slope
- Tangent

matrials

- Scientific calculator
- Computer graphing program

Think and discuss

1 Figure (1) shows the curve $f :]\, a\, ,\, b\, [\longrightarrow \mathbb{R}$ where $y = f(x)$, $\overleftrightarrow{CD}$ intersect the curve at the two points C $(x_1\, , f(x_1))$, D $(x_1 + h, f(x_1 + h))$. find the slope of the transversal $\overleftrightarrow{CD}$.

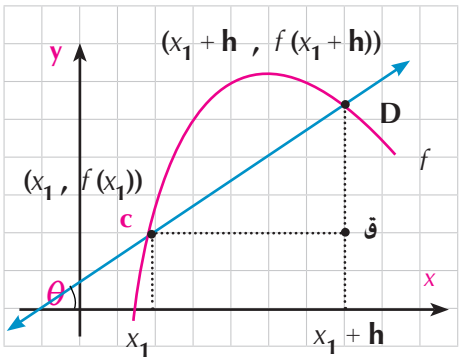


figure (1)

2 Consider x changes from x_1 to $x_1 + h$ compare the average rate of change and the slope of the transversal $\overleftrightarrow{CD}$. Is the following relation true? the slope of the secant $\overleftrightarrow{CD} = \tan\theta = \dfrac{f(x_1 + h) - f(x_1)}{h}$ $= A(h)$

3 If the point C $(x_1, f(x_1))$ is fixed on the curve of the function f, and the point D is moving on the curve approaching the point C so $\overleftrightarrow{CD}$ take the situation on $\overleftrightarrow{CN}$ and become a tangent to the curve at the point C .

i.e. $h \longrightarrow 0$

find the slope of the tangent to the curve of f at c

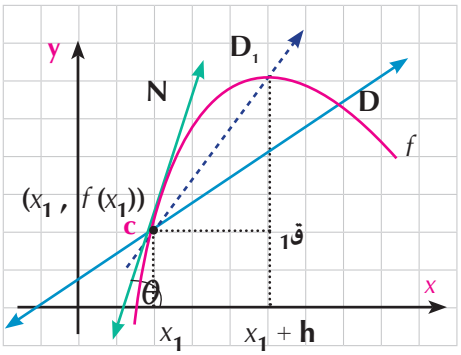


figure (2)

Notice :

the slope of the tangent at C $= \tan\theta = \lim\limits_{h \to 0} \dfrac{f(x_1 + h) - f(x_1)}{h}$ if exist

i.e. :

the slope of the tangent to the curve of the function f where $y = f(x)$ at the point $(x_1, f(x_1))$ equals the rate of change of f at $x = x_1$

Example

1 Find the slope of the tangent to the curve of the function f where $f(x) = 3x^2 - 5$ at the point A(2, 7) then find the measure of the positive angle which the tangent makes with the positive direction of the x-axis at the point A to the nearest minute .

Solution

$\because f(2) = 3(2)^2 - 5 = 7$ $\therefore$ the point A (2, 7) lies on the curve of f

the slope of the tangent at $(x = 2)$ = the rate of change of f at $(x = 2)$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\therefore \text{ the slope of the tangent} = \lim_{h \to 0} \frac{3(2+h)^2 - 5 - 7}{h} = \lim_{h \to 0} \frac{12h + 3h^2}{h}$$

$$= \lim_{h \to 0} (12 + 3h) = 12$$

then $\tan\theta = 12$ $\therefore \theta = \tan^{-1}(12) \simeq 85^\circ 14'$

Try to solve

1 Find the slope of the tangent to the curve of the function f where $f(x) = x^3 - 4$ at the point A (1, - 3) then find the measure of the positive angle which the tangent makes with the positive direction of the x-axis at the point A to the nearest minute .

Learn

The Derivative Function

Each value of the variable x in the domain of f corresponds a unique value of the rate change of f so the rate of change is also a function in the variable x called "the derivative function" or : " the first derivative of the function" or " the first differential coefficient "

Definition

If $f :]a, b[\longrightarrow \mathbb{R}$, $x \in]a, b[$ then the derivative function f' :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided this limit exists.

The derivative function symbols :

If $y = f(x)$ then the first derivative of the function f by the one of the symbols

y' or f' and read as " y derivative" or " f derivative "

$\frac{dy}{dx}$ and read as " dy by d x " or " the derivative of y with respect to x"

Notice that the slope of the tangent to the curve of $y = f(x)$ at the point $(x_1, f(x_1))$ is $f'(x_1)$

Example

(2) Using the definition of the derivative. Find the derivative of the function f where $f(x) = x^2 - x + 1$ then find the slope of the tangent at the point $(-2, 7)$

Solution

$\because f(x) = x^2 - x + 1$

$\therefore f(x + h) = (x + h)^2 - (x + h) + 1 \quad = x^2 + 2x\,h + h^2 - x - h + 1$,

$f(x + h) - f(x) = (2x + h - 1)\,h$

$\because f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ $\qquad \therefore f'(x) = \lim_{h \to 0} \frac{(2x + h - 1)\,h}{h}$

$f'(x) = \lim_{h \to 0} (2x + h - 1)$ $\qquad f'(x) = 2x - 1$

$\because f(-2) = (-2)^2 - (-2) + 1 = 7$ $\qquad \therefore$ the point $(-2, 7)$ lies on the curve of f

The slope of the tangent at the point $(-2, 7) = f'(-2) = 2(-2) - 1 = -5$

Try to solve

(2) Using the definition of the derivative to find the derivative of the function f when $f(x) = 3x^2 + 4x + 7$, then find the slope of the tangent to the curve at the point $(-1, 6)$.

Learn

Differintiability of a Function

The function f is said to be differentiable at $x = a$ (a belong to the domain of the function) if and only if $f'(a)$ exist where $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$

Example

(3) By using the definition, find the derivative of the function f where $f(x) = \sqrt{x - 1}$ then find $f'(5)$

Solution

$\because f(x) = \sqrt{x - 1}$ $\qquad \therefore$ the domain $f = [1, \infty[$

$\therefore f(x + h) = \sqrt{x + h - 1}$

$f(x + h) - f(x) = \sqrt{x + h - 1} - \sqrt{x - 1}$

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{\sqrt{x + h - 1} - \sqrt{x - 1}}{h}$ multiplying by the conjugate of the numerator

$$f'(x) = \lim_{h \to 0} \frac{(x + h - 1) - (x - 1)}{h(\sqrt{x + h - 1} + \sqrt{x - 1})}$$

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{x + h - 1} + \sqrt{x - 1})} = \frac{1}{2\sqrt{x - 1}} \quad , x > 1$$

notice that f is not differentiable at $x = 1$ because of the non-existance of the limit

$$f'(5) = \frac{1}{2\sqrt{5 - 1}} = \frac{1}{2 \times 2} = \frac{1}{4}$$

Try to solve

3 By using the definition , find the derivative of the function f where $f(x) = \sqrt{x + 5}$

Exercises 3 - 2

1 Find the derivative function of the function f in each of the following :

a $f(x) = 5x + 2$

b $f(x) = 3x^2$

c $f(x) = x^3 - 1$

d $f(x) = x^2 + 2x$.

2 Find the first derivative of the function f in each of the following and identify the values of x at which the function is not differentiable:

a $f(x) = \frac{1}{x}$

b $f(x) = \frac{1}{x + 3}$

c $f(x) = \frac{3}{2x - 5}$

d $\sqrt{x - 4}$

3 Find the derivative of the function f where $f(x) = x^3 + 4$ then find the slope of the tangent to the curve at the point (-1, 3) which lies on the curve .

4 Find the derivative of the function f where $f(x) = a\,x + b$ at any point (x, y) where a , b $\in$ R.

5 Find the slope of the tangent to the curve of the function f where $f(x) = 3x^2 - 8$ at the point A (2, 4) then find the measure of the positive angle which the tangent makes with the positive direction of x-axis .

6 If $f(x) = a\,x^2 + b$ where a and b are constants . Find :

a The first derivative of the function f at any point (x, y).

b The values of a , b if the slope of the tangent at the point (2, -3) which lies on the curve is 12.

Unit Three

3 - 3 Rules of Differentiation

We will learn

- The derivative of the constant function
- The derivative of $f(x) = x^n$
- The derivative of $f(x) = x$
- The derivative of $f(x) = ax^n$
- The derivative of the sum of two functions an the difference between them
- The derivative of product of two functions
- The derivative of quotient of two function
- The derivative of the composite function (the chain rule)
- The derivative of $y = (f(x))^n$.

Key - term

- First Derivative
- Product
- Quotient
- Chain Rule

matrials

- Scientific calculator.
- Computer graphing program.

Explore

1 - Using the definition, find the derivative of each of the following :

$f(x) = x^3$ $\qquad$ $f(x) = x^5$

2 - Can you discover the derivative of $f(x) = x^7$ without using the definition?

3 - Can you conclude a rule to derivative f where $f(x) = x^n$?

Learn

Derivative of a Function

1 - Derivative of the constant function

If $y = c$ **where:** $c \in \mathbb{R}$ $\qquad$ **then:** $\frac{dy}{dx} = 0$

Notice that :

$y = f(x) = c$, $f(x + h) = c$

$\because \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\therefore \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{c - c}{h} = 0 \ (h \neq 0)$

2 - the derivative of the function $f(x) = x^n$

If $y = x^n$ **where:** $n \in \mathbb{R}$ **then:** $\frac{dy}{dx} = n x^{n-1}$

If $y = x$ **then:** $\frac{dy}{dx} = 1$

If $y = a x^n$ **where:** $a, n \in \mathbb{R}$ **then:** $\frac{dy}{dx} = a n x^{n-1}$

Example

1 find $\frac{dy}{dx}$ for each of the following:

a $y = -3$ $\qquad$ **b** $y = x^4$ $\qquad$ **c** $y = 5x$

d $y = \frac{3}{x^2}$ $\qquad$ **e** $y = \sqrt{x^3}$

Solution

a $\because y = -3$ $\therefore \frac{dy}{dx} = 0$ $\qquad$ **b** $\because y = x^4$ $\therefore \frac{dy}{dx} = 4x^3$

c $\because\ y = 5x \qquad \therefore \frac{dy}{dx} = 5$

d $\because y = \frac{3}{x^2} = 3x^{-2} \qquad \therefore \frac{dy}{dx} = -6x^{-3}$

e $\because y = \sqrt{x^3} = x^{\frac{3}{2}} \qquad \therefore \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$ where $x \geqslant 0$

Try to solve

1 Find $\frac{dy}{dx}$ for each of the following:

a $y = x^7$ **b** $y = x^{10}$ **c** $y = x^{\frac{3}{2}}$ **d** $y = x^{-4}$

e $y = x^{-\frac{5}{3}}$ **f** $y = \frac{1}{x^5}$ **g** $y = \sqrt{x^9}$ **h** $y = \sqrt[3]{x^7}$

i $y = 3x^6$ **j** $y = -2x^{-5}$ **k** $y = \pi x^{-4}$ **l** $y = -4\sqrt[3]{x}$

The derivative of the sum or difference of two functions

If g and h are two differentiable functions with respect to the variable x, then $g \pm h$ is also differentiable with respect to x and $\frac{d}{dx}(g \pm h) = \frac{dg}{dx} \pm \frac{dh}{dx}$ and in general:

If $f_1, f_2, \ldots, f_n$ **are differentiable functions with respect to the variable** x **then:**

$$\frac{d}{dx}(f_1 \pm f_2 \pm f_3 \pm \ldots \pm f_n)(x) = f_1'(x) \pm f_2'(x) \pm f_3'(x) \pm \ldots \pm f_n'(x)$$

Example

2 Find $\frac{dy}{dx}$ in each of the following:

a $y = 2x^6 + x^{-9}$ **b** $y = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

Solution

a $\because\ y = 2x^6 + x^{-9}$

$\therefore \frac{dy}{dx} = 12x^5 - 9x^{-10}$

b $\because\ y = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

$= 1 - 2\sqrt{x} = 1 - 2x^{\frac{1}{2}}$

$\therefore \frac{dy}{dx} = 0 - 2 \times \frac{1}{2}x^{\frac{1}{2}-1} = -x^{-\frac{1}{2}}$

Try to solve

2 Find $\frac{dy}{dx}$ If:

a $y = 3x^2 + 5$ **b** $y = 2x^3 - 4x + 7$ **c** $y = 3x^{-4} + 6x^{\frac{2}{3}}$

d $y = 7x^4 - \frac{1}{4x}$ **e** $y = 3x^8 - 2x^{-5} + 8$ **f** $y = \frac{5}{x} + x\sqrt{x} - 4$

The derivative of the product of two functions:

If g and h are two differentiable functions with respect to the variable x then the function $(g \cdot h)$ is also differentiable with respect to the variable x and $\frac{d}{dx}(g \cdot h) = g \frac{dh}{dx} + h \cdot \frac{dg}{dx}$

Example

(3) Find $\frac{dy}{dx}$ if $y = (x^2 + 1)(x^3 + 3)$ then find $\frac{dy}{dx}$ when $x = -1$

Solution

$\because y = (x^2 + 1)(x^3 + 3)$

$\therefore \frac{dy}{dx} = (x^2 + 1) \times 3x^2 + (x^3 + 3) \times 2x$

$= 3x^4 + 3x^2 + 2x^4 + 6x$

$= 5x^4 + 3x^2 + 6x$

When $x = -1$ $\quad \therefore \frac{dy}{dx} = 5(-1)^4 + 3(-1)^2 + 6(-1) = 2$

Try to solve

(3) Find $\frac{dy}{dx}$ if :

a $y = (2x + 3)(3x - 1)$

b $y = (2x + 5)^2$

c $y = \sqrt{x}\ (\sqrt{x} + 4)$

d $y = (\sqrt{x} - 1)(\sqrt{x} + 1)$

e $y = (4x^2 - 1)(7x^3 + x)$ then find $\frac{dy}{dx}$ at $x = 1$

The derivative of the quotient of two functions:

If g and h are two differentiable functions with respect to the variable x and $h(x) \neq 0$

then the function $(\frac{g}{h})$ is also differentiable with respect to the variable x and $\frac{d}{dx}(\frac{g}{h}) = \frac{h\frac{dg}{dx} - g\frac{dh}{dx}}{h^2}$

i.e. $(\frac{g}{h})' = \frac{hg' - gh'}{h^2}$

Example

(4) Find $\frac{dy}{dx}$ if $y = \frac{x^2 - 1}{x^3 + 1}$

Solution

$\because y = \frac{x^2 - 1}{x^3 + 1}$

$\therefore \frac{dy}{dx} = \frac{(x^3 + 1) \times 2x - (x^2 - 1) \times 3x^2}{(x^3 + 1)^2}$

$\therefore \frac{dy}{dx} = \frac{2x^4 + 2x - 3x^4 + 3x^2}{(x^3 + 1)^2}$

$= \frac{-x^4 + 3x^2 + 2x}{(x^3 + 1)^2}$

Try to solve

4 Find $\frac{dy}{dx}$ if:

a $y = \frac{2x-1}{x+1}$ b $y = \frac{2x+5}{1-4x}$ c $y = \frac{2x^2-1}{x+5}$

The derivative of the xomposite function (Chain rule)

Theorem

If $y = f(z)$ is differentiable with respect to the variable z , and $z = g(x)$ is differentiable with respect to the variable x then $y = f(g(x))$ is differentiable with respect to the variable x and: $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$

notice that y is a function of function of x

this theorem is known as the chain rule

Example

5 If $y = (x^2 - 3x + 1)^5$ **find** $\frac{dy}{dx}$

Solution

let $z = x^2 - 3x + 1$ $\therefore y = z^5$

y is differentiable with respect to z (polynomial of z) then $\frac{dy}{dz} = 5z^4$

also z is differentiable with respect to x (polynomial of x) then $\frac{dz}{dx} = 2x - 3$,

Applying the chain rule $\therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 5z^2 \times (2x - 3)$,

substituting by z $\therefore \frac{dy}{dx} = 5(x^2 - 3x + 1)^4 \times (2x - 3)$

Try to solve

5 If $y = (2x+3)^5$ **find** $\frac{dy}{dx}$

the derivative of the funciton $[f(x)]^n$

If $z = [f(x)]^n$ where f is differentiable with respect to the variable x , n is a real number ,

then: $\frac{dz}{dx} = n[f(x)]^{n-1} \times f'(x)$

Example

6 Find $\frac{dy}{dx}$ if

a $y = (6x^3 + 3x + 1)^{10}$ b $y = (\frac{x-1}{x+1})^5$

Solution

a $y = (6x^3 + 3x + 1)^{10}$

$\therefore \frac{dy}{dx} = 10(6x^3 + 3x + 1)^9 \times \frac{d}{dx}(6x^3 + 3x + 1) = 10(18x^2 + 3)(6x^3 + 3x + 1)^9$

$= 30(6x^2 + 1)(6x^3 + 3x + 1)^9$

b $y = (\frac{x-1}{x+1})^5$ $\therefore \frac{dy}{dx} = 5(\frac{x-1}{x+1})^4 \times \frac{(x+1)\times 1 - (x-1)\times 1}{(x+1)^2}$

$= 5(\frac{x-1}{x+1})^4 \times \frac{x+1-x+1}{(x+1)^2}$

$= \frac{10}{(x+1)^2} \times (\frac{x-1}{x+1})^4 = \frac{10(x-1)^4}{(x+1)^6}$

Try to solve

6 Find $\frac{dy}{dx}$ if a $y = (2x^3 - 4x + 1)^9$ b $y = (\frac{5x^2}{3x^2+2})^3$

Example

7 If $f(x) = \frac{1}{3}x^3 - 2x^2 + 5x - 4$ find the values of x which make $f'(x) = 2$

Solution

$f'(x) = \frac{1}{3} \times 3x^2 - 2 \times 2x + 5 \times 1$

$= x^2 - 4x + 5$

When $f'(x) = 2$ $\therefore x^2 - 4x + 5 = 2$ $\therefore x^2 - 4x + 3 = 0$

$\because (x-1)(x-3) = 0$ $\therefore x = 1$ or $x = 3$

Try to solve

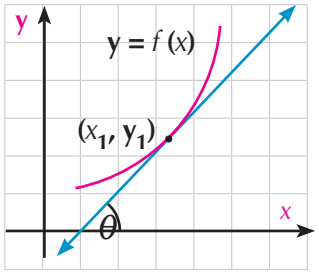
7 Find the values of x which make $f'(x) = 7$ in each of the following:

a $f(x) = x^3 - 5x + 2$ b $f(x) = (x-5)^7$

Critical thinking: Find $\frac{dy}{dx}$ if :

$y = (x-1)(x+1)(x^2+1)(x^4+1)(x^8+1)(x^{16}+1)$

Geometric Applications on the Derivative



In the geometrical explaining of the first derivative of the function f where $y = f(x)$ we found that the slope of the tangent (m) at the point (x_1, y_1) equals the first derivative of the function at that point.

i.e. $m = \frac{dy}{dx}$ at the point (x_1, y_1) and it can be written at the form

$$m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$$

and if θ is the measure of the positive angle which the tangent makes with the positive direction of x-axis, **then:**

$$m = \tan\theta = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$$

Notice that:

1 If m_1, m_2 are the slopes of two straight lines L_1, L_2 then :

$L_1 \parallel L_2$ if and only if $m_1 = m_2$ (the condition of parallelism)

, $L_1 \perp L_2$ if and only if $m_1 m_2 = -1$ (the condition of perpendicularity)

2 The slope of the tangent to a curve at any point on it is known as the slope of the curve at this point , also perpendicular to the tangent to a curve at the point of tangency is known as the normal to the curve at this point .

$\therefore$ the slope of the normal at the point $(x_1, y_1) = \frac{-1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$

Example

8 Find the slope of each of the tangent and the normal to the curve $y = 2x^3 - 4x + 5$ at the point (-2, -3) which lies on it.

Solution

$\because y = 2x^3 - 4x + 5$ $\therefore$ the slope of the tangent at any point $= \frac{dy}{dx} = 6x^2 - 4$

at the point (-2 , -3)

the slope of the tangent $= \left[\frac{dy}{dx}\right]_{(-2\,,\,-3)} = 6(-2)^2 - 4 = 20$

the slope of the normal $= \frac{-1}{\text{the slope of the tangent}} = \frac{-1}{20}$

Try to solve

8 Find the slope of each of the tangent an the normal to the following curves at the indicated points:

a $y = x - 7$ when $x = -1$

b $y = x + \frac{1}{x}$ when $x = 4$

c $y = \frac{5}{x - 3}$ when $x = 2$

d $y = (x^3 - 2)(x + 1)$ when $x = 1$

Example

(9) Find the points on the curve $y = x^2 - 6x + 5$ at which :

(a) the slope of the tangent = 2 (b) the tangent is parallel to x axis

(c) the tangent is perpendicular to the straight line $4y + x - 1 = 0$

Solution

$\because y = x^2 - 6x + 5$

$\therefore$ the slope of the tangent at any point $= \frac{dy}{dx} = 2x - 6$

(a) $\because \frac{dy}{dx} = 2 = 2x - 6 \quad \therefore 2x = 8$ i.e. $x = 4$

$\therefore y = (4)^2 - 6(4) + 5 = -3$

the slope of the tangent = 2 at the point (4, -3)

(b) $\because$ the tangent // x-axis $\therefore$ the slope of the tangent = the slope of x-axis = 0

$\frac{dy}{dx} = 2x - 6 = 0 \quad \therefore x = 3, y = 9 - 18 + 5 = -4$

the tangent at the point (3, -4) is parallel to x-axis

(c) the tangent is perpendicular to the straight line $4y + x - 1 = 0$ whose slope is $\frac{-1}{4}$

the slope of the tangent $= -1 \div \frac{-1}{4} = 4$

$2x - 6 = 4 \quad \therefore x = 5, y = 0$

$\therefore$ the tangent at the point (5, 0) is perpendicular to the straight line $4y + x - 1 = 0$

Remember

The slope of the straight line $ax + by + c = 0$ is $\frac{-a}{b}$

Try to solve

(9) Find the points which lie on the curve $y = x^3 - 3x^2$ at which the tangent to the curve :

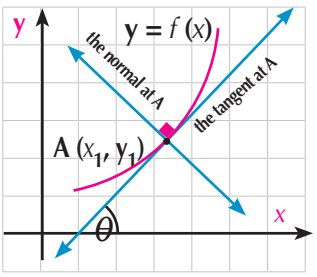
(a) Parallel to x-axis (b) perpendicular to the straight line $x + 9y + 3 = 0$

Learn

Equation of the Tangent to a Curve

If (x_1, y_1) is a point lie on the curve of the function f where $y = f(x)$, and m is the slope of the tangent to the curve at this point, then the equation of the tangent at the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$



Note: the equation of the normal to the curve at the point (x_1, y_1) is :

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

Example

10 Find the equations of the tangent and the normal to the curve $y = 2x^2 - 5x + 1$ at the point on it whose abscissa = 2

Solution

$\because$ the point lies on the curve $\therefore$ it satisfies its equation

at $x = 2$ then $y = 2(2)^2 - 5(2) + 1 = -1$ $\therefore$ the point is $(2, -1)$

$\because \frac{dy}{dx} = 4x - 5$ $\therefore (\frac{dy}{dx})_{x=2} = 4 \times 2 - 5 = 3$

i.e. at the point $(2, -1)$ the slope of the tangent = 3 and the slope of the normal = $\frac{-1}{3}$

$\therefore$ the equation of the tangent, the equation of the normal

$y - (-1) = 3(x-2)$ $y - (-1) = \frac{-1}{3}(x-2)$

$y + 1 = 3x - 6$ $3y + 3 = -x + 2$

$y - 3x + 7 = 0$ $3y + x + 1 = 0$

Try to solve

10 Find the equations of the tangent and the normal to the following curves at the indicated points:

a $y = x^3 + x^2 - 1$, $x = -2$ **b** $y = (3x - 5)^7$, $x = 2$

Example

11 If the straight line $y = ax^3 + bx^2$ touches the curve $y = 8x + 5$ at the point $(-1, -3)$ find the values of a , b.

Solution

$\because$ the point $(-1 , -3)$ lies on the curve $y = a x^3 + b x^2$

$\therefore -3 = a(-1)^3 + b(-1)^2$ i.e. $a - b = 3$ **(1)**

the slope of the tangent at any point on it = $\frac{dy}{dx} = 3 a x^2 + 2 b x$

$\because$ the straight line $y = 8x + 5$ touches the curve at the point $(-1, -3)$

$\therefore [\frac{dy}{dx}]_{(-1 , -3)}$ = the slope of the straight line = 8

$\therefore 3 a (-1)^2 + 2 b(-1) = 8$ i.e. $3 a - 2 b = 8$ **(2)**

Solving the two equations (1) and (2) **we get:**

$a = 2$, $b = -1$

Try to solve

11 Find the values of the constants a , b given that the tangent to the curve $y = x^2 + a x + b$ at the point $(1, 3)$ which lies on it is 5.

Exercises 3 - 3

Choose thr correct answer:

1. The rate of change of $x^3 + 4$ with respect to x when $x = 2$ equals

 a) 4 b) 8 c) $\frac{1}{4}$ d) 12

2. The slope of the tangent to the curve $y = 3x - x^3$ at $x = 0$ equals

 a) 3 b) zero c) - 3 d) 6

3. The tangent to the curve $y = x^2 - 8x + 2$ is parallel to x-axis at $x =$:

 a) -8 b) 2 c) 4 d) zero

4. The straight line $x + y = 5$ touches the curve $y = 3x^2 + 5x + 1$ at $x =$

 a) 1 b) 5 c) 3 d) -1

Complete the following:

5. $\frac{d}{dx}(2x) =$

6. $\frac{d}{dx}(3x^2 + 1) =$

7. $\frac{d}{dx}(x^4 - 2x^2 + 1)$

8. $\frac{d}{dx}(x + \sqrt{x})$

9. $\frac{d}{dx}(\frac{1}{x^3}) =$

10. $\frac{d}{dx}(5\pi) =$

11. $\frac{d}{dx}(x^{\frac{1}{2}}) =$

12. $\frac{d}{dx}(\frac{1}{\sqrt[3]{x}}) =$

13. $\frac{d}{dx}(5x^2 + 3x + 2) =$

14. $\frac{d}{dx}(\sqrt{2}\, x^7 - \frac{x^5}{5} + \pi) =$

15. Find $\frac{dy}{dx}$ for each of the following :

 a) $y = 3x^5$ b) $y = \frac{3}{4}x^{-8}$ c) $y = \frac{3}{2x^2}$ d) $y = 4\sqrt{x}$

 e) $y = \sqrt[3]{x^5}$ f) $y = \frac{2}{\sqrt{x}}$ g) $y = \frac{x^2}{\sqrt{x}}$ h) $y = x\sqrt{x}$

Find the first derivative with respect to x of each of the following.

16. $y = x^3 + 3x^2 - 5$

17. $y = \frac{1}{2}x^4 - \frac{2}{3}x^3 + 7x - 9$

18. $y = 2x^6 + 3\sqrt{x}$

19. $y = \frac{4x^2 - 3x + 2\sqrt{x}}{x}$

20. $y = x(3x^2 - \sqrt{x})$

21. $y = \frac{4x^2 - x + 3}{\sqrt{x}}$

22. $y = (x^2 + 1)(2x + 5)$

23. $y = (x^2 - 7)(2x + 3)$

24 $y = (x^2 + 3)(x^3 - 3x + 1)$

25 $y = (x^2 - \sqrt{x})(x^2 + 2\sqrt{x})$

26 $y = \frac{5x - 2}{5 \quad x + 1}$

27 $y = \frac{x - 2}{x + 5}$

28 Find the value of $\frac{dy}{dx}$ of each of the following at the indicated point:

a $y = (x^2 - 2)^7$ at $x = 0$

b $y = (x^2 - x + 1)^{-4}$ at $x = 1$

29 Find $\frac{dy}{dx}$ of each the following:

a $y = (x + 3)^7$

b $y = (2x2 - 3)^4$

c $y = (x^3 + x - 1)^5$

d $y^3 = 3x^2 + 1$

e $y = \sqrt[3]{(2x^3 - 4x + 7)^2}$

f $y = z^2$, $z = 3x^2 + 2$

30 Find the measure of the positive angle which the tangent to the curve $y = x^2 + \frac{1}{x} - 1$ makes with the positive direction of x-axis at $x = 1$.

31 Find the points on the curve $y = x^3 - 6x^2 - 15x + 20$ at which the tangent is parallel to x-axis.

32 Find the points on the curve $y = x^3 - 9x^2 - 16x + 1$ at which the slope of the tangent equals 5

33 Find the equations of the tangent and the normal to the curve $y = 3x^2 - 7x + 2$ at the point (2, 0) which lies on it .

34 Find the equation of the tangent to each of the following curves at the points on it whose abscissa is shown in the front of it:

a $y = (x + 3)^3$, $x = -1$

b $y = \frac{3}{x - 2}$, $x = 4$

c $y = \sqrt{x + 7}$, $x = 2$

d $y = (x - 5)(x + 5)$, $x = -3$

35 If the straight line $5x - y - 6 = 0$ touches the curve $y = a x^3 + b x$ at the point (1, -1). Find the values of a, b.

Unit Three

3-4 Integration

We will learn

- The anti derivative of a function.
- The indefinite integral
- Integral of some algebraic functions.

Key - term

- Antiderivative
- Integraation

matrials

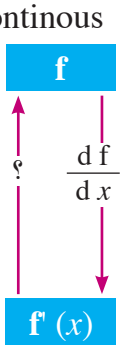
- Scientific calculator.
- Computer graphing program.

Explore

From the differentiation study, you have learned: the derivative of the function f where $f(x) = x^3 + c$, c is real constant is $f'(x) = 3x^2$.

i.e. $f'(x) = \frac{d}{dx} f(x)$

in this case the function f is called the original function of the function f'



In this lesson we will study the anti derivative which means, if we know the derivative function f'. How can we get the original function f ?

To find the original function whose derivative with respect to x is $5x^4$ let $f(x) = 5x^4$

Let's start with inverse way of differentiation operation

$n\, x^{n-1} = 5x^4$ $\quad\quad$ $\therefore n - 1 = 4$, $n = 5$

then $F(x) = x^5$ or $x^5 + 3$ or $x^5 - 2$

The function F is called the antiderivative of the function f

Can you discover the antiderivative F of function f if:

a $f(x) = 2x$ $\quad\quad$ **b** $f(x) = 7x^6$

Learn

Antiderivative

If $y = x^2$ **then the first derivative is** $\frac{dy}{dx} = 2x$

the conclusion of y from the derivative function $\frac{dy}{dx}$ is called integral operation or anti derivative.

for example x^2 is an anti derivative of the function $2x$ notice that $2x$ has many anti derivatives such $x^2 + 1$, $x^2 + 2$, $x^2 - 3$, ... the derivative of all of them is $2x$ and the constant c is the different between them .

$\therefore \frac{d}{dx}(x^2 + c) = 2x$ where (c) is constant.

Definition

The function F is called antiderivative of the function f if $F'(x) = f(x)$ for all x in the domain of f.

Example

(1) Prove that the function F where $F(x) = \frac{1}{2}x^4$ is antiderivative of the function f where $f(x) = 2x^3$.

Solution

Find the derivative of the function F then $F'(x) = \frac{1}{2} \times 4x^3 = 2x^3$

$\therefore F'(x) = f(x)$ **then the function** (F) is antiderivative of the function f

Try to solve

(1) Show that the function F where $F(x) = \frac{1}{2}x^6$ is antiderivative of the function f where $f(x) = 3x^5$

Critical thinking:

If each of F_1, F_2 is an antiderivative of the function f. What is the relation between F_1, F_2?

The indefinite Integral

The set of all antiderivatives of the function f is called the indefinite integral of this function and denoted by $\int f(x)\,dx$ [**is read as integral of** $f(x)$ **with respect to** x]

Definition

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$
where c is an arbitrary constant.

Notice that: $\frac{d}{dx}(x^3 + c) = 3x^2$ $\quad\therefore \int 3x^2\,dx = x^3 + c$

$\frac{d}{dx}(2x^7) = 14x^6$ $\quad\therefore \int 14x^6\,dx = 2x^7 + c$

To calculate the value of c we need to know the integral value at a specific value of the independent variable x and this is outside your study.

Example

(2) **Verify each of the following:**

a $\int x^7\,dx = \frac{1}{8}x^8 + c$

b $\int (7x^6 + \frac{4}{x^3})\,dx = x^7 - \frac{2}{x^2} + c$

Solution

a $\because \frac{d}{dx}\left(\frac{1}{8}x^8 + c\right) = x^7$ $\quad\therefore \int x^7\,dx = \frac{1}{8}x^8 + c$

b $\because \frac{d}{dx}\left(x^7 - \frac{2}{x^2} + c\right) = \frac{d}{dx}\left(x^7 - 2x^{-2} + c\right)$

$= 7x^6 - 2(-2)x^{-3} = 7x^6 + \frac{4}{x^3}$

$\therefore \int \left(7x^6 + \frac{4}{x^2}\right) dx = x^7 - \frac{2}{x^2} + c$

Try to solve

2 **Verify each of the following:**

a $\int x^{-4}\,dx = \frac{-1}{3}x^{-3} + c$ $\quad$ **b** $\int x\sqrt{1+x^2}\,dx = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$

Example

3 Using the principle definition of integration, find the antiderivative of the function f where:

a $f(x) = 5x^4$ $\quad$ **b** $f(x) = 18x^{\circ}$ $\quad$ **c** $f(x) = -3x^{-4}$

Solution

a $F(x) = \int 5x^4\,dx = x^5 + c$

b $F(x) = \int 18x^{\circ}\,dx = \int 3(6x^5)\,dx = 3x6 + c$

c $F(x) = \int -3x^{-4}\,dx = x^{-3} + c$

Try to solve

3 Using the definition of integration, find the anti derivative of each of : $20x^4$, $-5x^{-4}$

Finding the anti derivatives of functions by using the previous definition requires a lot of time and effort. So some integral standard forms can be used which facilitate the process of finding the anti derivative.

Rule (1) :

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \quad \textbf{where} \quad \textbf{c is constant , n is a rational number , } n \neq -1$$

Example

Find:

4 **a** $\int x^5\,dx$ $\quad$ **b** $\int x^{-3}\,dx$ $\quad$ **c** $\int x^{\frac{2}{5}}\,dx$ $\quad$ **d** $\int \frac{1}{\sqrt[5]{x^3}}\,dx$

Solution

a $\int x^5\,dx = \frac{x^{5+1}}{5+1} + c = \frac{1}{6}x^6 + c$

b $\int x^{-3}\,dx = \frac{x^{-3+1}}{-2} + c = -\frac{1}{2}x^{-2} + c$

c $\int x^{\frac{2}{5}}\,dx = \frac{1}{\frac{7}{5}}x^{+1\frac{2}{5}} + c$

$= \frac{5}{7}x^{\frac{7}{5}} + c$

d $\int \frac{1}{\sqrt[5]{x^3}}\,dx = \int x^{\frac{-3}{5}}\,dx$

$= \frac{1}{\frac{2}{5}}x^{1+\frac{3}{5}} + c = \frac{5}{2}x^{\frac{2}{5}} + c$

Try to solve

4 Find:

a $\int x^8\,dx$ **b** $\int x^{\frac{2}{3}}\,dx$ **c** $\int x^{\frac{7}{9}}\,dx$ **d** $\int \sqrt[7]{x^5}\,dx$

Rule (2) :

$\int a\,.f(x)\,dx = a\int f(x)\,dx$ **where** f is a constant real number

Example

5 **a** $\int 3x^3\,dx = 3\int x^3\,dx = 3\times\frac{1}{4}x^4 + c = \frac{3}{4}x^4 + c$

b $\int 8x^{-5}\,dx = 8\int x^{-5}\,dx = 8\times\frac{1}{-4}x^{-4} + c = -2x^{-4} + c$

Corollary

$\int a\,dx = a\,x + c$

then:

$\int 5\,dx = 5x + c$, $\int -9\,dt = -9t + c$

$\int dx = x + c$, $\int \sqrt{7}\,dz = \sqrt{7}\,z + c$

Try to solve

5 Find each of :

a $\int 3x^4\,dx$ **b** $\int -2z\,dz$ **c** $\int -x\,dx$ **d** $\int -\frac{dx}{5}$

Rule (3) :

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Example

6 Find: **a** $\int (4x + 3x^2)\, dx$ **b** $\int \frac{x^2 - 2x - 8}{x + 2}\, dx$

Solution

a $\int (4x + 3x^2)\, dx$

$= \int 4x\, dx + \int 3x^2\, dx$

$= 4\int x\, dx + 3\int x^2\, dx$

$= \frac{4}{2}x^2 + 3 \times \frac{1}{3}x^3 + c$

$= 2x^2 + x^3 + c$

b $\int \frac{x^2 - 2x - 8}{x + 2}\, dx$

$= \int \frac{(x - 4)(x + 2)}{x + 2}\, dx$

$= \int (x - 4)\, dx$

$= \frac{1}{2}x^2 - 4x + c$

Try to solve

6 Find:

a $\int (3x^2 + 2x - 1)\, dx$ **b** $\int (\frac{1}{x^2} + \sqrt{x} + 3)\, dx$

c $\int 2x(x + 3)\, dx$ **d** $\int \frac{4x^2 - 9}{2x - 3}\, dx$

Rule (4) :

$$\int [(ax + b)^n\, dx = \frac{1}{a(n + 1)}(ax + b)^{n+1} + c\ ,\ n \neq -1$$

Critical think:

1- Can you verify the previous rule by using the definition of anti deteriorative ? Explain.

Example

7 Find:

a $\int (3 - 2x)^5\, dx$ **b** $\int (2x - 7)^{-3}\, dx$

c $\int \frac{7}{\sqrt{3x - 4}}\, dx\ ,\ x > \frac{4}{3}$

Solution

a $\int (3x-2)^5\,dx = \frac{1}{3(5+1)}(3x-2)^{5+1}+c = \frac{1}{18}(3x-2)^6+c$

b $\int (2x-7)^{-3}\,dx = \frac{1}{2(-3+1)}(2x-7)^{-3+1}+c = -\frac{1}{4}(2x-7)^{-2}+c$

c $\int \frac{7}{\sqrt{3x-4}}\,dx = \int 7(3x-4)^{-\frac{1}{2}}\,dx = \frac{7}{3(-\frac{1}{2}+1)}(3x-4)^{-\frac{1}{2}+1}+c$

$= \frac{7}{3(-\frac{1}{2}+1)}(3x-4)^{\frac{1}{2}} = \frac{14}{3}\sqrt{3x-4}+c$

Try to solve

7 Find: **a** $\int 9(4-3x)^2\,dx$ **b** $\int \frac{15}{(3x-5)^6}\,dx$

Exercises 3 - 4

Find:

1 $\int x^2\,dx$ 2 $\int x^7\,dx$ 3 $\int 8x\,dx$

4 $\int -4x^3\,dx$ 5 $\int 9x^8\,dx$ 6 $\int 12x^{-4}\,dx$

7 $\int 5\,dx$ 8 $\int (5\sqrt[3]{x}\,dx$ 9 $\int -3x^7\,dx$

10 $\int -\frac{8}{5}x^{-3}\,dx$ 11 $\int \frac{7}{3}t^6\,dt$ 12 $\int \frac{12}{5}f^5\,df$

13 $\int (x+1)\,dx$ 14 $\int (5-2t)\,dt$ 15 $\int (t^3-6t^4)\,dt$

16 $\int (x^3+x^2+x)\,dx$ 17 $\int 2(3x^2+7)\,dx$ 18 $\int (ax^2-bx+c)\,dx$

19 $\int x(x+3)\,dx$ 20 $\int 7x^2(x^4-1)\,dx$ 21 $\int (x-1)(x+1)\,dx$

22 $\int (x-2)(2-x)\,dx$ 23 $\int (2x+3)(x-1)\,dx$ 24 $\int x^2(2x+\frac{1}{x})\,dx$

25 $\int \frac{3x^2-4x}{x}\,dx$ 26 $\int \frac{x^7+5x^6-x^3}{x^3}\,dx$ 27 $\int \frac{x^2-1}{x-1}\,dx$

28 $\int \frac{x^3-27}{x-3}\,dx$ 29 $\int \frac{x^3+8}{x^2-2x+4}\,dx$ 30 $\int \frac{x^2-5x+6}{x-3}\,dx$

31 $\int (x+4)^3\,dx$ 32 $\int 7(2x-7)^6\,dx$ 33 $\int (8-3t)^4\,dt$

34 $\int 6(x-3)^{-4}\,dx$ 35 $\int (2x-3)^{\frac{7}{3}}\,dx$ 36 $\int 15\sqrt{(3x-2)^5}\,dx$

Unit summary

Function of variation & rate of change

- If $y = f(x)$ and x varies from x to $x + h$ then the function of variation $V(h) = f(x+h) - f(x)$
- The average rate of change function $A(h) = \frac{f(x+h) - f(x)}{h}$
- The rate of change function $= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- The rate of change is called the first derivative of the function and is denoted by the symbol $f'(x)$ where :

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ [the first derivative is denoted by one of the symbols : $\frac{dy}{dx}$ or y' or $f'(x)$]

Differentiability:

- The function $f(x)$ is differentiable at $x = a$ if and only if: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exist.

Some rules of derivative:

The function $f(x)$	the derivative $f'(x)$
1- a where a is constant	zero
2- x^n where $n \in \mathbb{R}$	$n\,x^{n-1}$
3- ax^n where a constant , $n \in \mathbb{R}$	$a\,n\,x^{n-1}$
4- $g(x) \pm h(x) \pm \ldots k(x)$	$g'(x) \pm h'(x) \pm \ldots k'(x)$
5- $g(x) \cdot h(x)$ where g, r are differentiable	$g(x) \times h'(x) + h(x) \times g'(x)$
6- $\frac{g(x)}{h(x)}$ where g, r are differentiable $h(x) \neq 0$	$\frac{h(x) \times g'(x) - g(x) \times h'(x)}{(h(x))^2}$
7- If $y = f(z)$, $z = g(x)$	$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$
If $y = f(g(x))$	$f'(g(x)) \times g'(x)$

Geometrical applications:

1- If $y = f(x)$ then the slope of the tangent to the curve at any point on it $= \frac{dy}{dx}$.

2- If (x_1 , y_1) is a point on the curve of the function $y = f(x)$ then the equation of the tangent at the point (x_1 , y_1) is : $y - y_1 = m(x - x1)$ where $m = (\frac{dy}{dx})$ the slope of the tangent at point (x_1 , y_1) and the equation of the normal is $y - y_1 = \frac{-1}{m}(x - x_1)$

The antiderivative of a function:

- We say that the function F is antiderivative of the function f if $F'(x) = f(x)$ for all x belong to the domain of f.
- If $F'(x) = f(x)$ then $\int f(x)\, dx = F(x) + c$ where c is an arbitrary constant

Integration formulas

1- $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + c$ **where c is a constant , n is a rational number , n** $\neq -1$

2- $\int a \cdot f(x)\, dx = a \int f(x)\, dx$ **where a is a constant real number**

3- $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

4- $\int (ax + b)^n\, dx = \frac{1}{a(n+1)} (ax + b)^{n+1} + c$, $n \neq -1$

General exercises

1. If $f(x) = 5x - x^2$ find the variation function V at $x = x_1$ then find:
 - **a** The variation function V(h) when $x_1 = 1$
 - **b** The magnitude of the variation of the function when x_1 changes from 2 to 2.01.
 - **c** The magnitude of the variation of the function when x_1 changes from 1 to 0.98
2. If $f(x) = x^2 - 3x$ find :
 - **a** The average rate of change function at $x = 2$
 - **b** The average rate of change when x changes from 2 to 2.03
3. Find the function of average rate of change of the function f where $f(x) = \frac{1}{x}$ then calculate the rate of change of this function at $x = \sqrt{3}$
4. Find the function of average rate of change of the function f where $f(x) = x^2 - 2x + 3$ hence find:
 - **a** The average rate of change when x changes from -2 to -1.9
 - **b** The rate of change of this function when $x = 3$
5. Find the function of the average rate of change of the function f where $f(x) = \sqrt{x}$, then calculate the rate of change of this function when $x = 4$
6. A squared lamina expands regularly and maintaining its squared shape calculate the rate of change of the lamina's area with respect to the length of its side when the length of its side = 25 cm.
7. A circular lamina expands regularly and maintaining its circular shape . Find the rate of change of the lamina's area with respect to the length of its radius when the length of its radius is 14 cm, ($\pi = \frac{22}{7}$)

8 A cube expands regularly preserving its shape, calculate the rate of change of the volume of the cube with respect to the length of its edge when the length of its adge is 5cm.

9 A cube expands regularly preserving its shape, calculate the average rate of change of the total surface area of the cube with respect to the length of its side when the length of its side changes from 6 to 6.3 cm, then calculate the rate of change of the volume of the cuve when the length of its side is 8cm.

10 Find $\frac{dy}{dx}$

a $y = x^2 - 3x$ **b** $y = 5x^3 + 3x^2 - 4x + 7$

c $y = 2x^4 - \frac{5}{3}x^3 + 7x - 5$ **d** $y = 2x^3 - \frac{3}{x} - \frac{1}{x^2} + 5$

11 Find the slope of the tangent to the curve of each of the following functions:

a $y = x^2 - 2x + 3$ at the point (1, 2) **b** $y = x^3 - 2x + 1$ at the point (-1 , 2)

c $y = x^3 + x^2 - x + 1$ at the point (1 , 2) **d** $y = x - \frac{1}{x}$ at $x = 1$

12 Find the measure of the angle which the tangent of each of the following curves makes with the positive direction of x-axis:

a $y = x^2 - 3x + 2$ at (2 , 0) **b** $y = x^3 - x^2 - 4$ at (1 , -4)

c $y = (x- 1)(2x + 1)$ at the point (0, -1) **d** $y = x(x - 1)(x + 9)$ at the point (0 , 0)

13 Find the first derivative of each of the following

a $y = \frac{3x - 2}{2x + 3}$ **b** $y = \frac{x^2 - 5}{x^2 + 1}$

14 Find the slope of the tangent to the curve $y = \frac{x}{x + 2}$ at the point (-1, -1) on it.

15 Find the slope of the tangent to the curve $y = \frac{x^2 + x + 1}{x^2 + 1}$ at $x = 1$

16 Find $\frac{dy}{dx}$:

a $y = z^3$, $z = 2x + 1$ **b** $y = z^5$, $z = x^3 + 1$

c $y = z^4$, $z = 3x^2 - x + 2$ **d** $y = z^4$, $z = 2x^3 + 5$

Find the following integrals:

17 $\int (x^2 - 3x - 1)\,dx$ **18** $\int (\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}})\,dx$ **19** $\int (x - 2)(x + 2)\,dx$

20 $\int (x - 5)(x + 1)\,dx$ **21** $\int \frac{x^2 - 1}{x - 1}\,dx$ **22** $\int \frac{x^3 - 27}{x - 3}\,dx$

23 $\int 6(x - 2)^5\,dx$ **24** $\int 3(x + 3)^{-4}\,dx$

Accumulative test

Choose the correct answer:

1) if the average rate of change of $f = 3.2$ when x changes from 7 to 7.2 then the variation of f = ...

a) -6.4 b) 6.4 c) 1.6 d) -1.6

2) $\frac{d}{dx}(2 - 3x)^{-2}\, dx$

a) $12x^{-2} - 27x^{-4}$ b) $\frac{1}{3}(2 - 3x)^{-1}$ c) $6(2 - 3x)^{-3}$ d) $-2(2 - 3x)^{-3}$

3) The slope of the tangent to the curve of the function $y = (2x - 3)^5$ at $x = 2$ equals

a) 1 b) $\frac{1}{12}$ c) 5 d) 10

4) $\int (x^2 - 3)\, dx =$

a) $2x$ b) $x^3 - 3x$ c) $\frac{1}{3}x^3 - 3x + c$ d) $2x - 3 + c$

Answer the following:

5) Find the first derivative of each of the following functions:

a) $y = 3x^2 - 5x + 2$ b) $y = x(x^2 - 3x + 5)$

c) $y = (x - 2)(x + 2)$ d) $y = \frac{x - 1}{x + 1}$

6) Find $\frac{dy}{dx}$ of each of the following :

a) $y = z^2$, $z = x^4 - 6$ b) $y = 7z^3$, $z = \frac{1 + x}{1 - x}$

7) Find :

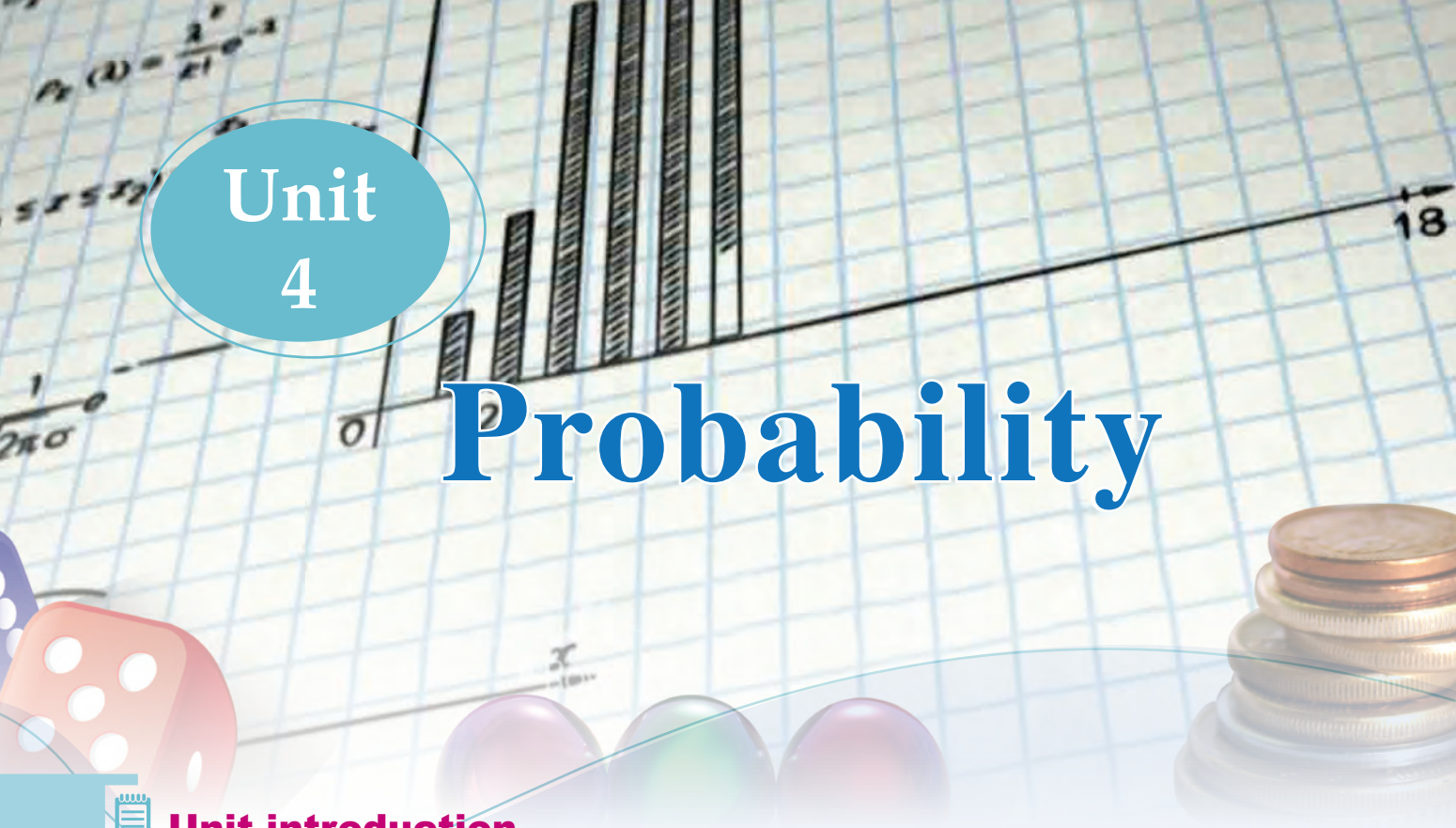
a) $\int (5x^4 - 3x^2 + 3)\, dx$ b) $\int 2x(x^2 + 3)\, dx$

c) $\int (2x + 3)^5\, dx$ d) $\int \frac{x^3 - 8}{x - 2}\, dx$

8) Find the equation of the tangent to the curve $y = (x - 2)(x + 1)$ at the points of intersection of the curve with x-axis.

9) Find the points on the curve $y = x^3 - 6x^2 - 15x + 20$ at which the tangent is parallel to x-axis.

10) Find the measure of the positive angle with the tangent to the curve $y = x^2 + \frac{1}{x} - 1$ makes with the positive direction of x-axis at $x = 1$.



Unit 4

Probability

Unit introduction

The roots of the science of probability are dept. to the revolution age from the studies of the scientists to space science, chance games and their trails to understand and analyze of the appearance of some elements from a large set of another elements in which (Gerolame Cardano) had done in the sixteenth century. And what (Pierr de Fermat) and (Blaise Pascal) lived in the seventeenth century.

In the way of which the science of probability improved and progress many definitions of the probability appear ,some is simple and depend on the sensitive reorganization ,some depend on the experimental method and the idea of the proportional repeating for the event we need to select by repeating the experiment several times under a fixed conditions. Probability is used to measure the capability of the occurrence of a certain event.

Starting from the ninetieth century, the probability theorems was discovered which considered as the largest aid to the statistical work .the scientists (Laplace) is one of its establishers and also the scientists (Adolph Quteelet) who present the first statistical work in a scientific way at 1853 .starting from this date statistics and probability take their places as valued science ,its great value in different fields and also in the scientific researched in different fields.

It passes all of this by determine a correct calculations in which we can depend on in predicting in the coming aspects. In this unit we will deal with Basic terms and concepts in probability and how to calculate it.

Unit objectives

By the end of the unit the students should be able to:

- Recognize the concept of the random experiment.
- Recognize the concept of the sample space.
- Write the sample space for some random experiments.
- Recognize the concept of the event, the simple event, the sure event and the impossible event.
- Recognize the concept of the mutually exclusive events.
- Recognize the operations on the events as (union, intersection ,difference ,complement)
- Recognize the concept of the probability.
- Use the axioms of probability to determine the probability of the occurrence of an event.
- Solve applied questions using the axioms of probability.
- Solve life applications problems using the laws of probability.

Key terms

- Statistics
- Probability
- Random Experiment
- Sample space
- Coin
- Die
- Event
- Simple Event
- Compound Event
- Certain Event
- Impossible Event
- Operation on the Events
- Mutually Exclusive Events

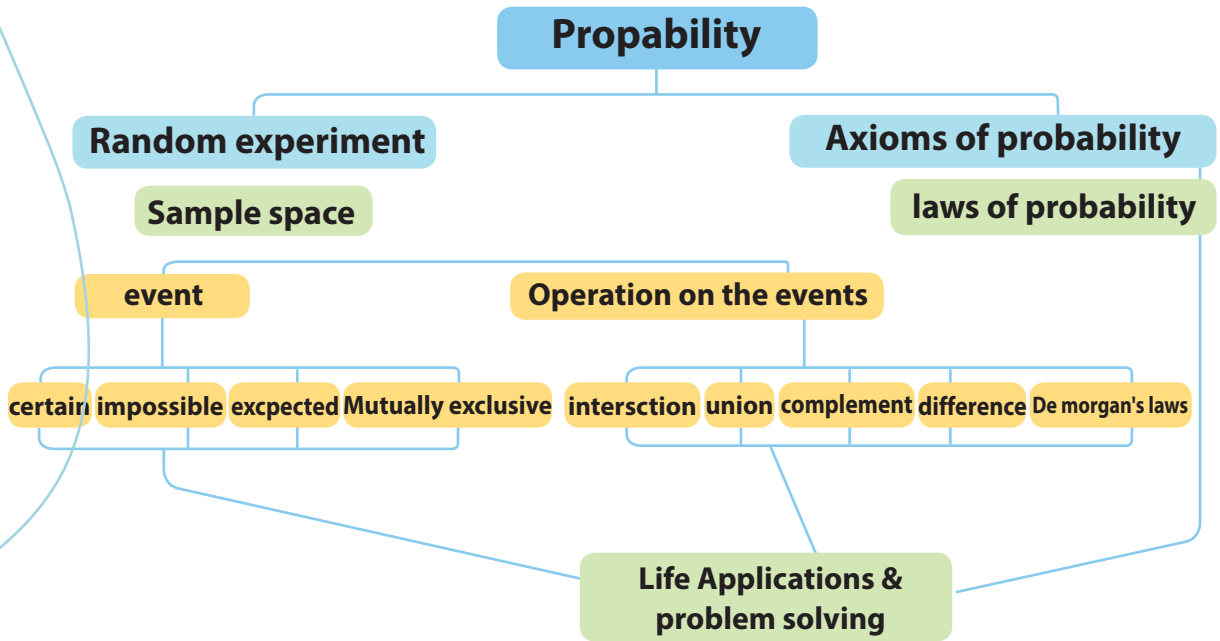
Unit lessons

Lesson (4 - 1): Calculating probability .

Materials

- Scientific calculator
- Graphical calculator
- Graphical programs

Unit planning guide



Unit 4

4 - 1 Calculating Probability

We will learn

- The concept of the random experiment and the sample space.
- The concept of the event –simple event- sure event-impossible event.
- Operations on events (union - intersection - difference - complement)
- Mutually exclusive events .
- De' Morgan's laws.
- Concepts of probability
- Calculating probability
- Probability axioms and its life applications.

Key - term

- random experiment
- sample space
- event
- simple event
- certain event
- impossible event
- mutually exclusive events
- probability
- probability axioms

Matrials

- Scientific calculator.

Introduction:

In our previous study, we learned the probability in a simple way. So we will complete the study of these concepts and the operations on events while calculating the probability of the occurrence of an event through examples and different life applications.

Basic terms and concepts

Learn

The random experiment: It's an experiment we know all of its outcomes before we do it but we cannot predict which of these outcomes will occur when we do the experiment.

Example

1 Show which of the following experiments represent a random experiment?

a Rolling a regular die and observe the number written in its upper face.

b Draw a color ball from a bag including a set of color balls (without determine their color) and recognize the color of the drawn ball.

c Throw a coin and observe what appears in its upper face.

d Draw a ball from a bag including four balls identical in volume and weight. The first is white, the second is black, the third is red and the fourth is green. Recognize the color of the drawn ball.

Solution

The experiments (a),(c),(d) are random experiments because we know all of their outcomes before we do each of them but we cannot predict which of these outcomes will occur when we do these experiments.

The experiment (b) is not a random experiment because we cannot determine the outcomes of the experiment before we do it.

Try to solve

1 Show which of the following experiments represent a random experiment?

a Throw a coin twice and observe the sequence of heads and tails.

b Draw a numbered card from a bag contains a set of numbered cards (we do not know their numbers) and recognize the written number on the drawn card.

c Draw a card from a bag contains a set of 20 identical cards numbered from 1 to 20 and observe the written number on the drawn card.

Learn

definition

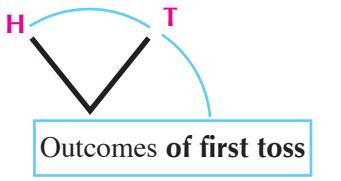
Sample space (outcomes space)

- The sample space of a random experiment is the set of all possible outcomes for this experiment, it denoted by (S)

Remarks:

- The number of elements in the sample space of a random experiment is denoted by n(S).
- The sample space will be finite if the number of its elements is finite and it will be infinite if the number of its elements is infinite. We will deal with the finite sample space.

The sample space for some famous random experiments:



First: Tossing a coin

1- The sample space of the experiment of tossing a coin once and observing the shown face is : S = { H, T}

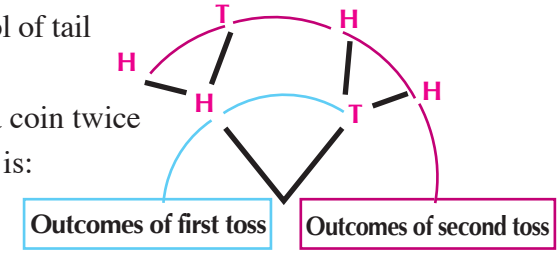
Where: H is the symbol of head, T is the symbol of tail

Where: n(S) = 2

2- The sample space of the experiment of tossing a coin twice and observing the sequence of heads, and tails is:

S = { (H, H), (H, T), (T, H), (T, T)}

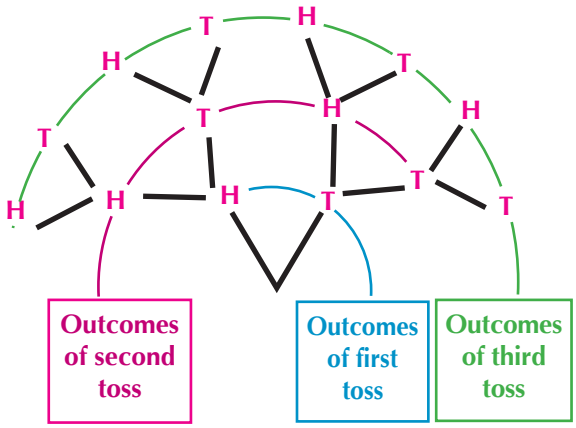
Where: n(S) = $2 \times 2 = 4 = 2^2$



3- The sample space of the experiment of tossing a coin three respective times and observing the sequence of heads and tails (could be as shown in the opposite tree diagram) is:

S = { (H , H , H) , (T , T , T) ,
(H , H , T) , (T , T , H) ,
(H , T , H) , (T , H , T) ,
(H , T , H) , (T , H , H)}

Where: n(S) = $2 \times 2 \times 2 = 8 = 2^3$



Note that

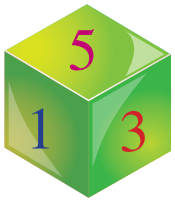
1- On tossing a piece of coin m times, then n (S) = 2^m

2- (H, T) ≠ (T, H) why?

3- The sample space of the experiment of tossing two different coins (different in shape and volume) simultaneously (at the same time) is the same sample space of tossing a coin two successive times, and each result of the experiment results is written in the form of an ordered pair (the face of the 1st coin, the face of the 2nd coin).



Second: Rolling a die



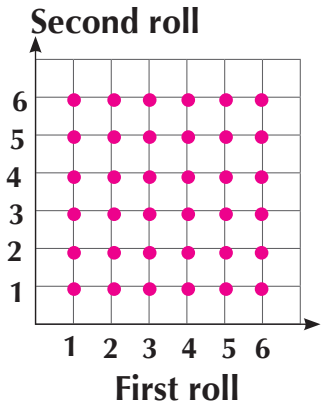
1- The sample space of the experiment of rolling a die once and observing the number shown on the upper face is:
S = { 1, 2, 3, 4, 5, 6 } where: n(S) = 6

2- The sample space of the experiment of rolling a die two successive times and observing the number shown each time on the upper face is the group of ordered pairs. The first co-ordinate is the result of the first roll and the second co- ordinate is the result of the second roll. i.e.: S = { (x, y) : x∈ { 1, 2, 3, 4, 5, 6}, y ∈ { 1, 2, 3, 4, 5, 6}} and the following figures illustrate this.

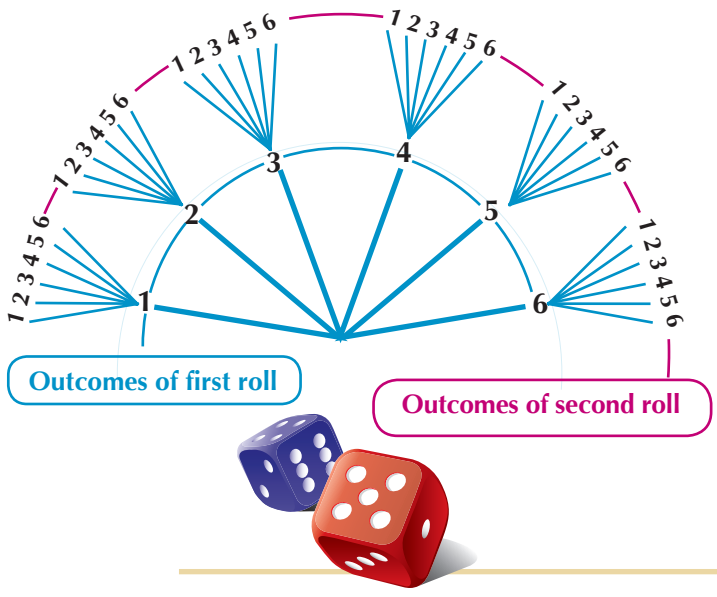
a Tabulated representation:

Toss: First / Second	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

b Geometrical representation:



c Tree diagram



Note that:

1- $n(S) = 6 \times 6 = 36 = 6^2$

2- $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

3- The sample space of the experiment of rolling two different dies at the same time (simultaneously) is the same sample space of rolling a die two successive times.

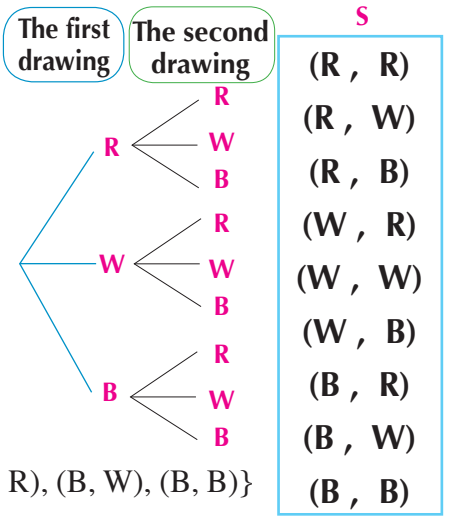
Example

(2) A bag contains three identical balls: the first is red, the second is white and the third is yellow. Write down the sample space if you draw two balls, one after the other, while reputing the drawn ball before drawing the other one (with replacement) and observing the succession of colors.

Solution

Denoting the red ball with the letter (R), the white ball with the letter (W) and the yellow ball with the letter (B):

First : When the ball is returned back to the bag before the second ball is drawn, each ball has the chance of appearance in the second drawing and it is then possible to draw the same ball twice. The opposite fig. shows the tree diagram of the sample space where n (S) = $3^2 = 9$



S = {(R, R), (R, W), (R, B), (W, R), (W, W), (W, B), (B, R), (B, W), (B, B)}

Add to your information

If a ball is drawn without replacement, that means: not to reputing the drawn ball before drawing the other one ,so there is no possible chance for that ball to appear in the second drawing.

Try to solve

(2) A box contains three identical balls numbered from 1 to 3. Two balls are drawn one after another with replacement and observing the number on the ball. Write down the sample space and find the number of its elements.

Learn

definition

The event

- An event is a subset from the sample space.

The simple event

- Is a subset from the sample space that contains only one element.

definition

The certain event

It is an event whose elements are the elements of the sample space S.
And it is an event that must occurs in each time we do the experiment.

The impossible event

It is an event that has no elements and is denoted by the symbol ϕ
And it is an event that must not occur each time we do the experiment.

Example

3 In the experiment of throwing a coin several times, the experiment will stop if a head or three tails appear.

Write down the sample space, and then determine the following events:

A The appearance of head at most

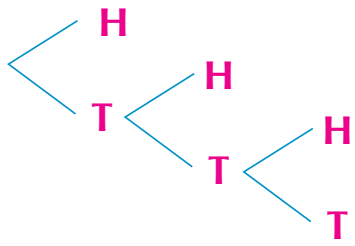
B The appearance of head at least

C The appearance of two tails at least

D The appearance of two heads at least

Solution

From the drawing, we find:



S = {H , (T , H),(T , T , H),(T , T , T)}

A = {H,(T , H),(T , T , H),(T , T , T)} = S

B = {H,(T , H),(T , T , H)}

C = {(T , T , H) , (T , T , T)}

D = { } = ϕ the impossible event

Try to solve

3 In the experiment of throwing a coin several times, the experiment will stop if two heads or two tails appear.

Write down the sample space, and then determine the following events:

A The appearance of head at least

B The appearance of two tails at most

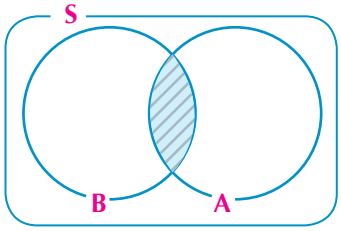
C The appearance of tail at most

Operation of the events

Learn

First: Intersection

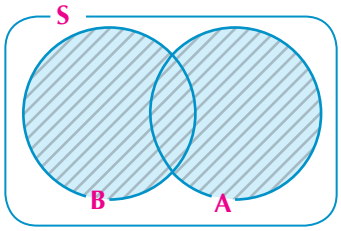
The intersection of the two events A and B is the event A ∩ B which contains all elements of the sample space that belong to both A and B and **means the occurrence of A and B (the occurrence of the two events at the same time).**



A ∩ B

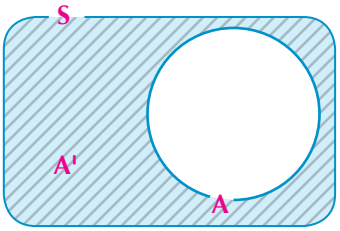
Second: Union

The union of the two events A and B is the event A ∪ B which contains all elements of the sample space that belong to A or B or both of them and **means the occurrence of A or B (the occurrence of one of them at least).**



A ∪ B

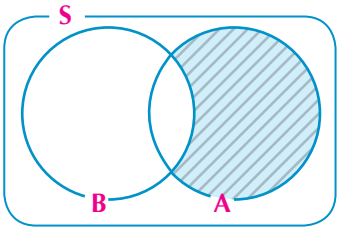
Third: Complement



The event A` is called the complementary event of the event A, where A` contains all elements of the sample space that does not belong to the event A, and **means non - occurrence of the event A** .

Note : $A \cup A' = S$, $A \cap A' = \phi$

Fourth: Difference



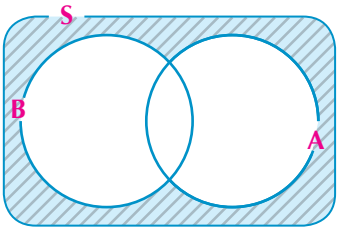
The event A - B contains all elements of the sample space that belong to A and does not belong to B it also contains the same elements of the event $A \cap B'$

and it means the occurrence **of A and non-occurrence of B (occurrence of A only).**

$A - B = A \cap B' = A - (A \cap B)$

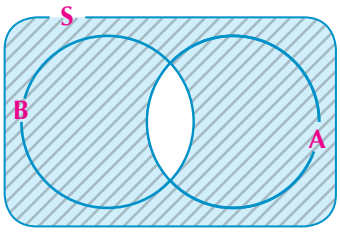
Fifth: De morgan's laws

A and B are the two events from the sample space S, then:



(first) $A' \cap B' = (A \cup B)'$

which means the event of (non-occurrence of any of the two events) or the event of (the non-occurrence of A and the non-occurrence of B)



(second) $A' \cup B' = (A \cap B)'$

which means the event of (non-occurrence of any of the two events all together) or the event of (the occurrence of one of the events at most)

Learn

Mutually exclusive events

Two events A and B are mutually exclusive events if the occurrence of one of them prevent the occurrence of the other

For example: **1-** If A" event of success in an exam" , B "event of failure in the same exam", then The occurrence of one of them prevent the occurrence of the other.

2- In experiment of rolling a die once, observing the number on the upper face

then $S = \{1, 2, 3, 4, 5, 6\}$

If A: appearance of an odd number $A = \{1, 3, 5\}$

B: appearance of an even number $B = \{2, 4, 6\}$

then $A \cap B = \phi$ so, the occurrence of one of them prevent the occurrence of the other.

definition

- Two events A and B are said to be mutually exclusive if $A \cap B = \phi$
- Several events are said to be mutually exclusive if and only if each two by two are mutually exclusive events.

Notice that :

1- If $A \cap B = \phi$, then A and B are mutually exclusive events.
If A, B and C are three events in a sample space S and: $A \cap B = \phi$, $B \cap C = \phi$, $C \cap A = \phi$ then: A , B, C are said to be mutually exclusive events and vice versa.

2- Simple events (primary) in any random experiment are mutually exclusive.

3- Any event A and its complement A` are mutually exclusive events.

Example

(4) Two distinct dice are tossed and observing the numbers on the upper faces.

First: represent the sample space geometrically, and then write down the following two events.

A "appearance of the same numbers on the two faces"

B "appearance of two numbers their sum equals 7"

Second : Are A , B mutually exclusive? Justify your answer

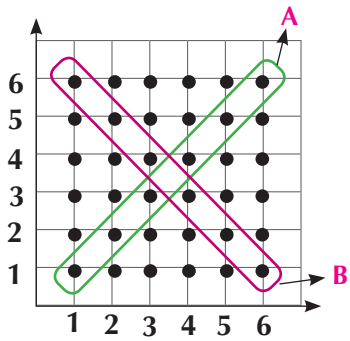
Solution

First : The elements of the sample space are ordered pairs, their number = $6^2 = 36$

the opposite figure is the geometrical representation of the sample space where every element is represented by a point

A = {(1, 1), (2, 2), (3, 3), (4, 4) , (5, 5) , (6, 6) }

B = { (6, 1) , (5, 2), (4, 3) , (3, 4) , (2, 5) , (1, 6) }



Second: $\because A \cap B = \phi$ $\therefore$ A, B are mutually exclusive events

Try to solve

(4) In the previous example: write each of the following events:

C "appearance of two numbers their sum equal 5"

D "appearance of two numbers one of them is twice the other"

Are C , D mutually exclusive? Justify your answer.

Propability

Learn

Calculation of probability :

If A is an event in the sample space S for a random experiment all its outcomes (primary events) are equal possibility i.e. $A \subset S$, number of elements of event A equals n (A), number of elements of S equals n(S), if we denoted the probability of occurrence of A by P(A):

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{The number of the outcomes leads to the occurence of the event A}}{\text{The number of all possible outcomes for the random experiment}}$$

Example

5 If a ball drawn from a box contains 10 identical balls, 5 of them are white, 2 are red and the rest are green .Find the probability of the following events:

A the event "the drawn ball is red"

B the event "the drawn ball is red or green"

C the event "the drawn ball is not green"

Solution

The probability that the drawn ball is red = $P(A) = \frac{\text{The number of red balls}}{\text{The number of all balls}} = \frac{2}{10} = 0.2$

The probability that the drawn ball is red or green = $\frac{\text{The number of red balls + green balls}}{\text{The number of all balls}}$

$= \frac{2+3}{10} = \frac{5}{10} = 0.5$

The probability that the drawn ball is not green = P(C)

= The probability that the drawn ball is red or white = $\frac{2+5}{10} = 0.7$

Think: Can you obtain P(C) with another method? Explain that.

Try to solve

5 In the previous example: find the probability of the following events:

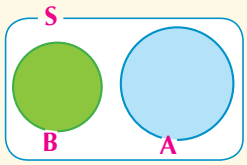
D: the event "the drawn ball is red or green"

E: the event "the drawn ball is red or white or green"

Learn

Axioms of probability

1- For every event $A \subset S$ there exists a real number called probability of event A, and denoted by P(A) Where : $0 \leqslant P(A) \leqslant 1$



2- $P(S) = 1$

3- If $A \subset S$, $B \subset S$

and A , B are mutually exclusive events, then : $P(A \cup B) = P(A) + P(B)$

From the previous axioms we notice that:

The first axiom means that the probability of the occurrence of any event is a real number belongs to the interval [0, 1]

The second axiom means that the probability of the sure event = 1

The third axiom is called the sum of probabilities of mutually exclusive events rule which is circulating for a finite number of mutually exclusive events.

$P(A_1 \cup A_2 \cup A_3 \cup ... \cup A_n) = P(A_1) + P(A_2) + P(A_3) + ... + P(A_n)$

where $A_1, A_2, A_3, ..., A_n$ are two by two mutually exclusive events

Important Corollaries

(1) $P(\phi) = 0$

(2) $P(A') = 1 - P(A)$

(3) $P(A - B) = P(A) - P(A \cap B)$

(4) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Add to your information

If $A \subset B$ then $P(A) \leqslant P(B)$

Example

6 If A , B are two events in a sample space S of a random experiment, such that:

$P(A) = \frac{3}{8}$, $P(B) = \frac{3}{4}$, $P(A \cap B) = \frac{1}{4}$, Find:

a $P(A \cup B)$ **b** $P(A')$ **c** $P(A - B)$ **d** $P(A' \cap B')$

Solution

a $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{8} + \frac{3}{4} - \frac{1}{4} = \frac{7}{8}$

b $P(A') = 1 - P(A) = 1 - \frac{3}{8} = \frac{5}{8}$

c $P(A - B) = P(A) - P(A \cap B) = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$

d $P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B) = 1 - \frac{7}{8} = \frac{1}{8}$

Try to solve

6 In the previous example, find the following probabilities:

a $P(B')$ **b** $P(B - A)$ **c** $P(A' \cup B')$

Example

7 If A , B are two events in a sample space S of a random experiment, such that $P(A) = \frac{5}{8}$, $P(B) = \frac{1}{2}$, $P(A - B) = \frac{3}{8}$ Find :

a $P(A \cap B)$ **b** $P(A \cup B)$ **c** $P(A' \cap B')$ **d** $P(A' \cup B)$

Solution

a $P(A \cap B) = P(A) - P(A - B) = \frac{5}{8} - \frac{3}{8} = \frac{2}{8} = \frac{1}{4}$

b $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{8} + \frac{1}{2} - \frac{1}{4} = \frac{7}{8}$

c $P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B) = 1 - \frac{7}{8} = \frac{1}{8}$

d $P(A' \cup B) = P(A \cap B')' = 1 - P(A \cap B') = 1 - P(A - B)$

$= 1 - \frac{3}{8} = \frac{5}{8}$

Think : Can you obtain $P(A' \cup B)$ with another method? Explain that.

Try to solve

7 In the previous question, find the following probabilities:

a $P(A')$ b $P(A' \cup B')$ c $P(B \cap A')$

Example

8 If A and B are two events in a sample space S of a random experiment, such that $P(A') = \frac{1}{3}P(A)$, $P(B) = \frac{1}{2}$, $P(A' \cup B') = \frac{5}{8}$ Find :

a The probability of occurrence of one of the two events at least.

b The probability of occurrence of one of the two events at most.

c The probability of occurrence of the event B only

d The probability of occurrence of only one of the two events.

Solution

$\because P(A' \cup B') = \frac{5}{8}$ $\therefore P(A \cap B)' = 1 - P(A \cap B) = \frac{5}{8}$ $\therefore P(A \cap B) = \frac{3}{8}$

$\because P(A') = \frac{1}{3}P(A)$ $\therefore 1 - P(A) = \frac{1}{3}P(A)$ $\therefore \frac{4}{3}P(A) = 1$ $\therefore P(A) = \frac{3}{4}$

a The probability of occurrence of one of the two events at least $= P(A \cup B)$
$= P(A) + P(B) - P(A \cap B) = \frac{3}{4} + \frac{1}{2} - \frac{3}{8} = \frac{7}{8}$

b The probability of occurrence of one of the two events at most $= P(A \cap B)'$
$= P(A' \cup B') = \frac{5}{8}$

c The probability of occurrence of the event B only $= P(B - A)$
$= P(B) - P(A \cap B) = \frac{1}{2} - \frac{3}{8} = \frac{1}{8}$

d The probability of occurrence of only one of the two events $= P(A \cup B) - P(A \cap B)$
$= \frac{7}{8} - \frac{3}{8} = \frac{1}{2}$

Think: Can you find the probability of occurrence of only one of the two events with another method? Explain this.

Try to solve

8 If A and B are two events in a sample space S of a random experiment, such that $P(A) = 0.8$, $P(B) = 0.6$, $P(A \cup B)' = 0.1$ Find The probability of the following events:

a The occurrence of one of the two events at least.

b The occurrence of the event A only

c The occurrence of only one of the two events

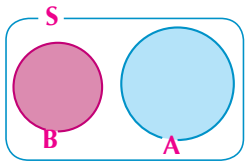
d The occurrence of one of the two events at most.

Example

9 A and B are two events in a sample space S of a random experiment, where:
$P(B) = 3P(A)$, $P(A \cup B) = 0.72$, find: $P(A)$, $P(B)$

First: if A , B are mutually exclusive events. **Second:** if $A \subset B$

Solution

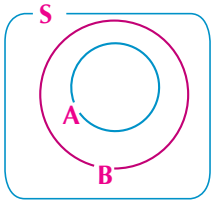


Let P (A) = x ∴ P(B) = 3 x

First: ∵ A , B are mutually exclusive events.

∴ P (A ∪ B) = P (A) + P (B) : 0.72 = 3 x + x

∴ x = 0.18, P (A) = 0.18 , P (B) = 0.54



Second: ∵ A ⊂ B ∴ A ∪ B = B

P (A ∪ B) = P (B) = 3x = 0.72

∴ P (A) = 0.24 , P (B) = 0.72

Try to solve

9 If A and B are two events in a sample space of a random experiment , Where:

$P(B) = \frac{1}{5}$, $P(A \cup B) = \frac{1}{3}$ Find P (A)

a If A , B are mutually exclusive events. b if B ⊂ A

Critical thinking:

Explain how to calculate P (A) if A ⊂ S, S is a sample space of a random experiment, if $\frac{P(A')}{P(A)} = \frac{3}{7}$

Try to solve

10 IF S is a sample space of a random experiment where S = {A , B , C}, and $\frac{P(A')}{P(A)} = \frac{2}{3}$, $\frac{P(B')}{P(B)} = \frac{5}{2}$ Find P(C)

Example

10 **Join with the school inviroment:** If the probability of success of a student in the physics exam equals 0.85 and the probability of success in mathematics exam equals 0.9 and the probability of success in both subjects equals 0.8. Find the probability of:

a The success of the student in at least one of the two subjects.

b The success of the student in mathematics only.

c The non-success of the student in both subjects together.

Solution

Let A denoted the event of success of the student in physics , B: denoted the event of success of the student in mathematics.

then: P (A) = 0.85 , P (B) = 0.9 , P (A ∩ B) = 0.8

a Probability of success of the student in at least one of the two subject = P (A ∪ B)

∴ P (A ∪ B) = P (A) + P (B) - P (A ∩ B) = 0.85 + 0.9 - 0.8 = 0.95

b Probability of success of the student in mathematics only means probability of success in mathematics and not success in physics P (B - A)

∴ P (B - A) = P (B) - P (B ∩ A) = 0.9 - 0.8 = 0.1

c The event of the student will not success in both subjects = $(A \cap B)'$ which is the complement of $(A \cap B)$

$\therefore P(A \cap B)' = 1 - P(A \cap B) = 1 - 0.8 = 0.2$

Life applications:

Try to solve

11 To get a job in a company, a person has to pass two exams: theoretical and practical. If the probability to succeed in the theoretical exam is 0.75, the probability to succeed in the practical exam is 0.6 and the probability to succeed in both of them is 0.5. If a person applies to this job for the first time. Find the probability of:

a Success in the theoretical exam only. b Success in at least one of the two exams.

Critical thinking:

Join with sport: A coach of one of the sports teams says in a news briefing that the probability that his team wins in the away match is (0.7), the probability of winning in the rematch is (0.9) and the probability of winning both matches is 0.5. Does the concept of probability agree with the words of the coach? Justify your answer.

Example

11 A die is rolled two consecutive time. The number on the upper face is observed in each time. Determine each of the following events:

First : A The appearance of two numbers their sum is less than or equal 4.

Second: B One of the two numbers is twice the other.

Third: C The asbsolute difference between the two numbers equals 2

Fourth: D The appearance of two numbers their sum is more than 12

Solution

$n(S) = 36$

First: $A = \{(1, 1), (1 , 2) , (1, 3), (2, 1), (2, 2), (3, 1)\}$ $\therefore$ $n(A) = 6$, $\therefore P(A) = \frac{6}{36} = \frac{1}{6}$

Second: $B = \{ (1, 2), (2, 1), (2, 4), (4, 2), (3, 6) , (6, 3)\}$ $\therefore$ $n(B) = 6$ $\therefore P(B) = \frac{6}{36} = \frac{1}{6}$

Third: $C = \{(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3) , (4, 6) , (6, 4)\}$ $\therefore$ $P(C) = \frac{8}{36} = \frac{2}{9}$

Fourth: Its impossible to get two numbers their sum is more than 12, $\therefore$ $D = \phi$, $P(D) = 0$

Try to solve

12 In the previous example, calculate the following probabilities:

First: A the event "the two appearance numbers are equal".

Second: B the event "the number in the first roll is odd and the number in the second roll is even"

Example

12 A coin is tossed three consecutive times, the sequence of heads and tails is observed. Find the probability of each of the following:

First: A appearance of only one head.

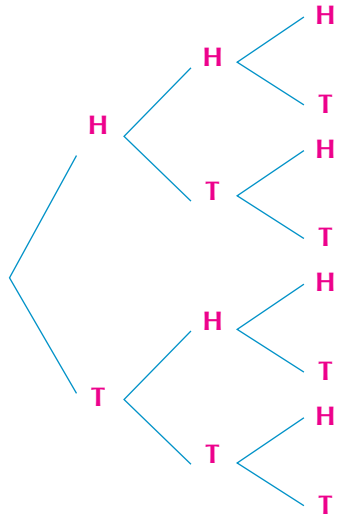
Second: B appearance of at least two heads.

Third: C appearance of exactly two heads.

Solution

$S = \{ (H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T)\}$,

$n(S) = 8$



First: $\because$ A appearance of only one head.

$\therefore A = \{ (H, T, T), (T, H, T), (T, T, H)\}$,

$\because n(A) = 3 \qquad \therefore P(A) = \frac{3}{8}$

Second : $\because$ B appearance of at least two heads means either two or more heads

$\therefore B = \{ (H, H, T), (H, T, H), (T, H, H), (H, H, H)\}$

$\because n(B) = 4 \qquad \therefore P(B) = \frac{4}{8} = \frac{1}{2}$

Third: $\because$ C appearance of exactly two heads

$\therefore C = \{ (H, H, T), (H, T, H), (T, H, H)\} \quad \because n(C) = 3 \qquad \therefore P(C) = \frac{3}{8}$

Try to solve

13 In the previous example, calculate the probability of:

First : A appearance of the same face in the three tosses **Second :** B appearance of at most one head.

Third: C appearance of odd number of heads **Fourth:** D appearance of at least one tail.

Fifth: E appearance of number of heads equals number of tails.

Example

13 **Join with community:** In one of the conferences, 200 persons from different nationalities participated in the conference and their data is represented by the following table:

	Speak Arabic	Speak English	Speak French	Total sum
Man	50	45	25	120
Woman	45	30	5	80
Sum	95	75	30	200

If one of the participants is chosen randomly, then find the probability that the chosen person is:

a A woman speaks Arabic.
b A man speaks English.
c Speaks Arabic or French.
d Speaks Arabic and English.
e A woman does not speak English and does not speak Arabic.

Solution

a The probability that the chosen person" A woman speaks Arabic" $= \frac{45}{200} = 0.225$

b The probability that the chosen person" A man speaks English" $= \frac{45}{200} = 0.225$

c The probability that the chosen person" Speaks Arabic or French" $= \frac{95 + 30}{200} = 0.625$

d The probability that the chosen person" Speaks Arabic and English" $= p(\phi) = 0$

e The probability that the chosen person" A woman does not speak English and does not speak Arabic" $= \frac{5}{200} = 0.025$

Try to solve

14 In the previous example, find the probability that the chosen person:

a Does not speak English.
b Speaks German.
c A woman speaks French or English.
d A man speaks Arabic or a woman speaks English.

Exercises (4 - 1)

1 A student wants to buy a bag. It is possible to choose from three types. Each one has two sizes and the color of the bag is either black or brown. Represent the sample space by a tree diagram.

2 In an experiment of tossing a coin once, then a die is rolled, observing the upper faces.

a Write down the sample space of this experiment, then determine the following events.

- A: appearance of a head and an odd number.
- B: appearance of a tail and an even number
- C: appearance of a prime number > 2
- D: appearance of a number divisible by 3

3 A die is rolled two consecutive times, the number on the upper face is observed in each time Determine each of the following events:

- A: The appearance of two equal numbers
- B: The appearance of two numbers their sum equals 9
- C: The appearance of two numbers their sum equals 13
- D: The appearance of number 3 one time at least.

4 From the set of numbers {1, 2, 3, 4} we need to form a two different digit number. Represent the sample space in a tree diagram, and then determine the following events:

- A: The event "the unit digit is an odd number ".
- B: The event "the tens digit is an odd number".
- C The event "the two digits are odd numbers".
- D The event "the unit digit or the tens digit is an odd number.

5 A bag contains 20 identical cards numbered from 1 to 20, If a card is selected randomly and the number written on it is recorded. Write the following events:

A the event "the recorded number is even and greater than 10

B the event "the recorded number is a factor of 12"

C the event "the recorded number is odd and divisible by 3

D the event "the recorded number is a multiple of the two numbers 2, 5

E the event "the recorded number is prime"

F the event "the recorded number satisfying the inequality $5x - 3 \leqslant 17$

6 Two cards are drawn one after the other from a set of 8 identical cards numbered from 1 to 8 and the drawn card must returned before drawn another card. What is the number of the elements in the sample space? and if:

A: is the event "the number in the second draw is three times the number in the first draw"

B: is the event "the sum of the two numbers is more than 13"

Write the events A, B. Are there two mutually exclusive events? Explain that.

7 In the experiment of tossing a coin three consecutive times and observing the sequence of heads and tails .represent the sample space with tree diagram, then determine the following events:

A the event "appearance of two tails at least" B the event "appearance of two tails at most"

C the event " appearance of a head in the first toss"

A the event " non-appearance of a head in the three tosses"

8 In an experiment of tossing a coin once, then a die is rolled, observing the upper faces Represent the sample space of this experiment by a probability tree diagram, and then determine the following events:

A appearance of a Tail and an even number"

B appearance of a head and an odd number"

C non-occurrence of A or non- occurrence of B"

D occurrence of the event A only

E occurrence of the event A and occurrence of the event B

Choose the correct answer from those given :

9 If a regular die is rolled once, then the probability of the appearance of an odd number less than 5 in the upper face equals:

a $\frac{2}{5}$ b $\frac{1}{2}$ c $\frac{1}{3}$ d $\frac{1}{6}$

10 If a regular die is rolled twice, then the probability of the appearance of an even number in the first roll and a prime number in the second roll equals:

a $\frac{1}{3}$ b $\frac{1}{6}$ c $\frac{1}{9}$ d $\frac{1}{4}$

11 If a ball is drawn randomly from a box contained 3 white balls, 5 red balls and 7 green balls, then the probability that the selected ball is white or green equals:

a $\frac{1}{5}$ b $\frac{2}{3}$ c $\frac{7}{15}$ d $\frac{1}{2}$

12 A card is drawn from a set of 9 identical cards numbered from 1 to 9. What is the probability that the drawn card carrying a divisor of (factor of) 9 or an odd number equals:

a $\frac{1}{3}$ **b** $\frac{7}{9}$ **c** $\frac{1}{2}$ **d** $\frac{5}{9}$

13 If A , B are two events in a sample space of a random experiment $B \subset A$,
$P(A) = 2P(B) = 0.6$ then $P(A - B)$ equals:

a 0.6 **b** 0.3 **c** 0.4 **d** 0.2

14 A uniform die, the numbers 8, 9, 10, 11, 12, 13 written in its faces. If the die is rolled once, observing the number appearing on its upper face

a Find the probability of each of the following events:

- A: "appearance of an odd number."
- B "appearance of a prime number."
- C: "appearance of an even number."
- D "appearance of a number great than 12."
- E: "appearance of a number consists of two digits."
- F "appearance of a number consists of only one digit."

b Calculate: $P(A \cup C)$, $P(E \cup F)$, $P(B \cap D)$.

15 If is a sample space of a random experiment, where S = {A , B, C, D}, find:
P(A) , P(B) , given that P (A) = 3 P(B), $P(C) = P(D) = \frac{7}{18}$

16 If A, B are two mutually exclusive events, S is a sample space of its random experiment, If $P(A \cup B) = 0.6$, $P(A - B) = 0.25$ find, P(A), P(B).

17 If A, B is a sample space of a random experiment, and $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{8}$, $P(A \cap B) = \frac{1}{4}$ find:

a P(A') **b** $P(A \cup B)$ **c** P(A - B) **d** $P(A' \cap B')$

18 If A, B are two events, of a sample space of a random experiment, where: P(A) = 0.4 , P(B') = 3P(B) , $P(A \cap B) = 0.2$ find the probability of:

a Occurrence of A only. **b** Occurrence of A or B
c Occurrence of A and non-Occurrence of B.

19 A box contains colored identical balls: 4 red, 6 blue, and 5 yellow. A ball is selected at random from the box, find the probability that the drawn ball is:

a Red. **b** Blue or yellow. **c** Not blue. **d** Not red and not yellow.

20 One card is selected at random from 30 identical cards numbered from 1 to 30. Find the probability that the selected card is carring a number:

a Divisible by 3 **b** divisible by 5
c Divisible by 3 and 5 **d** divisible by 3 or 5

21 Three distinct coins are tossed once. Observing the upper faces, find the probability of the following events:

- A: appearance of a head or two heads.
- B: appearance of at least one head.
- C: appearance of a head at most.
- D: appearance of at two consecutive tails at least.

22 A die is rolled two consecutive times, the number on the upper face is observed in each time. Find the probability of each of the following events:

- Appearance of the number 4 in the first roll.
- Appearance of two numbers, their sum equals 8
- Appearance of two numbers, their sum is less than or equal 5

23 **Join with sport :** A random sample consists of 60 persons in a survey, it is found that 40 of them encourage Al Hilal club, 28 of encourage El negma club and 8 of them don't encourage any of them. A person is chosen at random from the sample. Find the probability that the chosen person encourages:

a At least one of the two clubs.
b Both clubs.
c Al Hilal club only.
d Only one of the two clubs.

24 In an experiment of tossing a coin once, then a die is rolled once , observing the upper faces. If A is the event of the appearance of a head and a prime number, B is the event of the appearance of an even number. Find the probability of the occurrence of each of the two events, and then calculate the probability of the following events:

a The occurrence of one of the events at least
b The occurrence of the two events together
c The occurrence of only the event B
d The occurrence of only one of the two events

25 a card is selected randomly from 50 identical cards numbered from 1 to 50, if the number written on it is recorded. Find the probability that the number written on the selected card is:

a A multiple of number 7
b A perfect square number
c A multiple of number 7 and a perfect square number
d Not a perfect square number and not a multiple of 7

26 If A , B are two events, in a sample space of a random experiment, where: $P(B) = \frac{4}{5} P(A)$, $P(A - B) = 0.24$ $P(B \cap A') = 0.15$ then find: $P(A)$, $P(B)$, $P(A \cup B)$, $P(A' \cup B')$

27 Tarek wrote 75 letters on the typewriter ,he found that 60% of them are without mistakes and Zead wrote 25 letters on the typewriter ,he found that 80% of them are without mistakes. If a letter is selected randomly from all letters written by both Tarek and Zead ,then find the probability that the selected letter is:

a Without mistakes.
b Written by Zead.
c Written by Zead without mistakes.
d Written by Tarek with mistakes.

28 If A , B are two events, in a sample space of a random experiment, where: $P(A) = 0.6$, $P(B) = 0.8$, $P(A' \cup B') = 0.5$ then find $P(A' \cap B)$

unit Summary

1 **Random experiment:** It is an experiment in which we can specify in advance all its possible outcomes before carrying it, but we cannot predict in certainty which of these outcomes will exactly occur. When we do it.

2 **Sample space "Outcomes space":** It is the set of all possible outcomes of a random experiment. it is denoted by S

3 **An event :** It is a subset of the sample space.

4 **Simple event:** It is a subset of a sample space that contains a single element.

5 **Certain " sure" event:** It is the event whose elements are all of the sample space (S).

6 **Impossible event:** An event has no element , denoted by (ϕ).

7 **Operations on events:** Intersection - union - complement - difference.

8 **Mutually exclusive events**

- Two events A and B are mutually exclusive if $A \cap B = \phi$.
- Several events are said to be mutually exclusive events if each two by two events are mutually exclusive

9 **Calculations of probabilities**

- If S is the sample space of a random experiment ,the prime events on it are equal opportunities
- The probability of the occurrence of the event $A \subset S$ is denoted by P(A) such that $P(A) = \frac{n(A)}{n(S)}$

10 **Axioms of probability**

- For every event $A \subset S$ there exists a real number which is called probability of A and denoted by P where: $0 \leqslant P(A) \leqslant 1$
- $P(S) = 1$
- If $A \subset S$, $B \subset S$ and A, B are mutually exclusive events, then: $P(A \cup B) = P(A) + P(B)$

11 If $A_1 \cup A_2 \cup A_3 \cup\cup A_n = S$ where $A_1 , A_2 , A_3 , ... , A_n$ are mutually exclusive events

then $P(A_1) + P(A_2) + P(A_3) + ... + P(A_n) = 1$

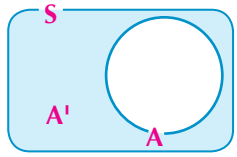
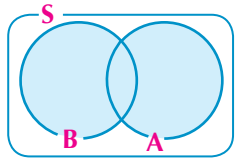
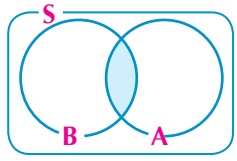
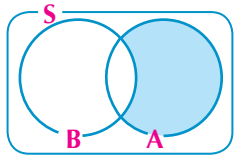
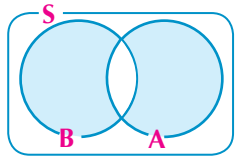
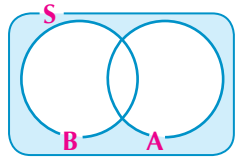
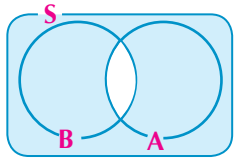
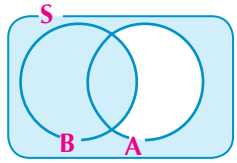
12 $P(\phi) = 0$

13 If $A \subset S$ where S is the sample space of a random experiment, then $P(A') = 1 - P(A)$

14 If A , B are two events in the sample space of a random experiment ,then:

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (b) $P(A - B) = P(A) - P(A \cap B)$

15 Events in a verbal form, their representations by venn diagram and their probabilities:

The event in a verbal description	Representation of the event by a Venn–	Probability of occurrence of the event
Non occurrence of event A		$P(A') = 1 - P(A)$
Occurrence of A or B (occurrence of at least one of them)	 $A \cup B$	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Occurrence of A and B (occurrence of both of them together)	 $A \cap B$	$P(A \cap B) = P(A) + P(B) - P(A \cup B)$
Occurrence of A only (occurrence of A and non occurrence of B)	 $A - B = A \cap B' = A - (A \cap B)$	$P(A - B) = P(A) - P(A \cap B) = P(A \cap B')$
Occurrence of only one of them (occurrence of A only or occ. of B only)	 $(A - B) \cup (B - A)$	$P((A - B) \cup (B - A))$ $= P(A - B) + P(B - A)$ $= P(A \cup B) - P(A \cap B)$
Non occ. of any of the two events. (Non occ. of event A and Non occ. of event B	 $(A \cup B)' = A' \cap B'$	$P(A \cup B)' = P(A' \cap B') = 1 - P(A \cup B)$
Non occ. of both of them together. (Non occ. of event A or Non occ. of event B) (The occ. of one of them at most)	 $(A \cap B)' = A' \cup B'$	$P(A \cap B)' = P(A' \cup B') = 1 - P(A \cap B)$
Non occurrence of A only (occ. of B or not occ. of A)	 $(A - B)' = B \cup A'$	$P(A - B)' = 1 - P(A - B) = P(B \cup A')$ $= P(A') + P(A \cap B)$

General Exercise

Choose the correct answer from those given:

In an experiment of rolling a die twice then:

1. The probability of the appearance of number 5 in the first roll and number 6 in the second roll is:

 a $\frac{1}{24}$ b $\frac{1}{30}$ c $\frac{1}{36}$ d $\frac{1}{6}$

2. The probability of the appearance of number 5 in one of the two rolls and the number 6 in the other roll is:

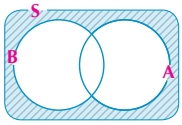
 a $\frac{1}{12}$ b $\frac{1}{6}$ c $\frac{5}{36}$ d $\frac{1}{18}$

3. The probability of the appearance of two equal numbers in the two rolls

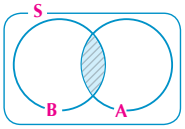
 a $\frac{1}{5}$ b $\frac{1}{36}$ c $\frac{1}{6}$ d $\frac{1}{18}$

If A and B are two events in a sample space of a random experiment, choose the event that represents the shaded parts in the opposite venn diagram :

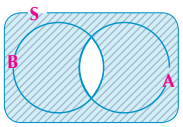
4. a $S - (A \cup B)$ b $A' \cup B'$
 c $S - (A' \cup B')$ d $(A \cap B)'$



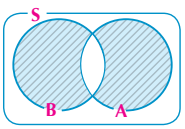
5. a $A \cap B$ b $A \cup B'$
 c $(A \cup B)'$ d $A' \cup B'$



6. a $(A \cup B) - (A \cap B)$ b $S - (A \cap B)'$
 c $S - (A \cup B)'$ d $S - (A \cap B)$



7. a $S - (A \cap B)$ b $A' \cap B'$
 c $(A - B) \cup (B - A)$ d $S - (A \cup B)'$



8. Form two digit number from the digits of the number 4321 , and represents the sample space using the probability tree diagram ,then write the sample space and the following events:
 A: event" the set of prime numbers"
 B: event" the set of numbers that divisible by 3"
 C: event" the set of numbers that divisible by 3 and 5"
 D: event" the set of numbers that its unit digit is twice of its tens digit"

9. In the experiment of rolling a die once and observing its upper face .Find the probability that the appearance number is:
 a A prime number b A factor of 6 c An odd number and divisible by 3

10 If A, B are two events in a sample space S of a random experiment.

Complete each of the following to get true statement:

a If A, B are two mutually exclusive events:

➢ $A \cap B =$ ➢ $A - B =$ ➢ $B - A =$

b If $B \subset A$ then:

➢ $(A \cap B) =$ ➢ $A \cup B =$ ➢ $B - A =$

c If $P(A) = 0.3$, $P(B) = 0.5$, $P(A \cap B) = 0.1$ then:

➢ $P(A \cup B) =$ ➢ $P(B - A) =$ ➢ $P(A' \cap B') =$

d If $A \cap B = \phi$, $P(A') = 0.7$, $P(B') = 0.4$ then:

➢ $P(A \cup B) =$ ➢ $P(A \cap B) =$ ➢ $P(A - B) =$

11 If A, B are two events in a sample space S of a random experiment, $A \subset B$, $P(A) = \frac{1}{2}$, and probability of occurrence of B only = 0.2. Calculate the probability of non-occurrence of B.

12 If A, B are two events in a sample space S of a random experiment , $P(A) = \frac{1}{2}$, , $P(B) = x$ and $P(A \cup B)' = \frac{1}{3}$

a Find the value of x in each of the following:

➢ If A, B are two mutually exclusive events. ➢ $A \subset B$.

b If $x = \frac{1}{4}$ Find the value of $P(A \cap B)$.

13 If A, B are two events, of a sample space S of a random experiment, write a symbolical expression of each of the following events, representing them by Venn -diagrams:

a Non occurrence of A

b Occurrence of A or B.

c Occurrence of B only.

d Occurrence of A or non-occurrence of B.

e Non occurrence of both of them together.

f Occurrence of only one of them.

14 A coin is tossed three consecutive times, observing the upper faces, represent the sample space by a tree diagram. Find the probability of the following events:

a A: appearance of only two heads

b B: appearance of at most two heads

c C: appearance of at most one tail

d D: appearance of same outcomes in the three tosses

15 A die is tossed two consecutive times, the number on the upper face is observed in each time.

a Draw a tree diagram that represents the sample space, explaining the following events on it.

- A: appearance of two numbers, their sum is an odd number which is greater than 6
- B: appearance of two numbers , one of them is 2 and their sum $\leqslant 5$
- C: appearance of two equal numbers

b Are A, B and C two by two mutually exclusive events?

c Calculate the probability of each of the following: $P(A \cup B)$, $P(B \cap C)$, $P(A \cup C)'$, $P(B - C)$.

16 Five identical cards numbered from 2 to 6, two cards are drawn randomly one after the other with replacement and observing the sequence of the numbers on the selected cards to form all possible two digit numbers. Find the probability that:

a its unit digit is a prime number.

b its tens digit is an odd number.

c its unit digit is a prime number or its tens digit is an odd number.

Newspaper:

17 In a sample of 50 persons, it is found that 27 of them read newspaper (A), 24 of them read newspaper (B) and 9 persons read both newspapers. A person is chosen at random from this sample. Find the probability that the chosen person read:

a Newspaper A only.

b At least one of the two newspapers.

Tourism:

18 In one of the sound and light shows at the pyramids, 200 people attend from various nationalities, the opposite table shows their data. If one of them was chosen at random by their access cards to give him the souvenir prize, then: Find the probability that the chosen person is :

	Arab	European	American	Total
Man	32	47	15	94
Woman	23	63	20	106
Total	55	110	35	200

a An European man.

b An American Woman.

c A Woman.

d With Arab or European nationality.

19 If S = {A, B, C}, is a sample space of a random experiment and, $20P(A) = 15P(B) = 12P(C)$ Find: $P(A)$, $P(B)$, $P(C)$

Accumulative Test

Complete the following:

1. In an experiment of rolling a die once and observing the number appearing on the upper face, then the sample space S = ...
2. In an experiment of tossing a coin twice and observing the upper face, then the event of the occurrence of a head at most = ...
3. In an experiment of rolling a die, then tossing a coin once and observing the upper face in each of them ,then the event of the appearance of a prime number = ...
4. In an experiment of rolling a die twice and observing the number appearing on the upper face each time, then the event that" the sum of the two numbers equals 5" = ...
5. If a card is drawn randomly from 20 identical cards numbered from 1 to 20, and its number is observed, then the event that" the appearance number is divisible by 3" = ...
6. In an experiment of tossing a coin three consecutive times and observing the appearance of the heads and the tails on its upper face, then the event of the appearance of "exactly two heads = ...
7. If $A \subset S$ where S is the sample space of a random experiment and $P(A') = 3\ P(A)$ find $P(A')$.
8. If a card is drawn randomly from 20 identical cards numbered from 1 to 20, and its number is observed, find the probability that the number on the drawn card:

 a Divisible by 6 b Prime number more than 10 c A factor of number 12
9. A and B are two events in a sample space for a random experiment, if $P(A \cup B) = 0.85$, $P(A) = 0.75$, $P(B') = 0.6$ find:

 a $P(A \cap B)$ b $P(A \cap B')$ c $P(A' \cup B')$
10. A and B are two events in a sample space for a random experiment, if $P(A) = \frac{2}{3} P(B)$, the probability of the occurrence of one of them at most equals 0.75 , the probability of the occurrence of one of them at least equals 0.6 find the probability of the following events:
 a The occurrence of both of them all together.
 b The occurrence of only one of them.
 c The occurrence of B or the non-occurrence of A.
11. A and B are two events in a sample space for a random experiment, if $P(A') = \frac{3}{5}$ $P(A \cup B) = 0.45$ find $P(B)$ in the following cases:

 a A and B are mutually exclusive events b $A \subset B$ c $P(B - A) = 0.2$

12 **Join with Tourism:** A tour consists of 19 tourists from Russia, 17 from italy, 14 tourists from France. One of them was chosen at random, calculate the probability that the tourist:

a From Russia or France.
b Not from France.
c From Europe.
d From Netherlands.

13 **Join with school environment :** In a school celebration for exceling students, if the probability that the governor attends is 0.8, and the probability that the General Director of Education attends is 0.9 and the probability that both of them attend together is 0.75 find:

a The probability that the governor only attends.
b The probability that at least one of them attends.
c The probability that both of them will not attend.

14 A and B are two events in a sample space S of a random experiment. If P(A) = 0.6, P(B') = 0.3, P(A ∪ B) = 0.9, find the probability of each of the following events :

a Occurrence of A or B
b Occurrence of A and non-occurrence of B
c Occurrence of A only or occurrence of B only

15 S is the sample space of a random experiment ,where S = {A, B, C}

if $\frac{P(A')}{P(A)} = \frac{7}{3}$, 2 P (B) = 3 P(B'), Find $\frac{P(C')}{P(C)}$

16 If A, B are two events in a sample space S of a random experiment, and P(A) = 0.6, P(B) = 0.5 , P (A'∪ B') = 0.7 Find: The probability of the following:

First: the occurrence of both of the two events together
Second: the occurrence of event A only
Third: the occurrence of at least one of the two events
Fourth: the occurrence of only one of the two events

17 50 persons request a job in one of the banks and their data is represented in the following table. If a person is selected randomly, find the probability that the selected person is:

First: female.

Second: With middle qualification.

Third: male with high qualification.

Fourth: female or with high qualification.

Gender	High qualification	Middle qualification	Sum
Male	16	14	30
Female	12	8	20
Sum	28	22	50

General tests

The first test — Algebra

Answer the following questions:

Question 1: choose the correct answer:

1) The rule of the sequence (1×2) , (2×3) , (3×4) , (4×5) , (5×6) is

a) $(n - 1)(n + 1)$ b) $n(n + 1)$ c) $2n(n + 1)$ d) $(n + 1)(n + 2)$

2) The arithmetic mear of the two numbers 8 and 12 is

a) 8 b) 12 c) 10 d) 2

3) The sum of the series $\sum_{r=1}^{3} 3r$ = ...

a) 1 b) 3 c) 6 d) 18

4) ${}^5P_3$ = ...

a) 5 b) 3 c) 60 d) 15

Question 2:

1) Find the order of the first negative term of the sequence (11 , 9 , 7 , ...)

2) In the arithmetic sequence $(16 + 40x , 50x - 9 , ... , 3x + 15 , 5x + 6)$ find the value of x , then find number of terms of the sequence .

Question 3:

1) A geometric sequence of positive terms if $T_1 + T_2 = 48$, $T_3 + T_4 = 192$

a) Write the sequence.

b) Find the sum of the first ten term of the sequence.

2) Expand each of the following series, then find the sum of the expansion . Verify your answer using calculator

a) $\sum_{r=2}^{7} 1 - 2r$ b) $\sum_{r=1}^{8} (\frac{-1}{2})^r$

Question 4:

1) Find the value of each of:

a) $\lfloor 5 - \lfloor 3$ b) $\lfloor 5 \times \lfloor 2$

2) Find the solution set of each of the following equations:

a) ${}^{12}C_r = {}^{12}C_{r^2+3}$ b) $\frac{\lfloor n}{\lfloor 8} = 1$

The second test — Algebra

Question 1: Choose the correct answer:-

1. The geometric mean of the two numbers 4 and 16 is
 a) 8 b) 8 c) ± 8 d) 4

2. The sum of the sequence (1 , 3 , 5 , ... , 17 , 19) is
 a) 20 b) 50 c) 100 d) 30

3. If the arithmetic mean of the two numbers x and 16 is 3 then x equals
 a) -10 b) 3 c) 16 d) 6

4. If $^5C_r = 10$ then r = ...
 a) 5 b) 3 c) 10 d) 1

Question 2:

1. Insert 8 arithmetic means between the two numbers 2 and 29 then find the sum of those means.

2. The sum of the first and the second terms of an infinite geometric sequence of positive terms is 36, if the square of its third term is 36. Find the sequence, then find its sum starting from its first term.

Question 3:

1. A geometric sequence of positive terms in which $T_2 = 12$, $T_4 = 2\,T_3$ find the sum of the first seven terms of this sequence.

2. Find the sum of each of the following serieses:
 a) $\sum_{r=1}^{10} (\frac{1}{3})^{r-1}$ b) $\sum_{k=3}^{9} 3k + 1$

Question 4:

1. Find the value of each of the following:
 a) $\lfloor 5 - \lfloor 2$ b) $\sum_{r=1}^{3} {}^4C_r$

2. How many ways can 5 persons arranged in a row?

The third test — Algebra

Answer the following questions:

Question 1: Choose the correct answer:-

1. A geometric sequence in which $T_n = \frac{1}{2} T_{n-1}$ then the common ratio of the sequence equals:

 a) $\frac{1}{2}$ b) 2 c) $\frac{3}{2}$ d) $\frac{2}{3}$

2. If $T_n = 15 - 3n$ then the first negative term of the sequence is:

 a) $_3T$ b) $_5T$ c) $_6T$ d) $_{15}T$

3. The seventh term of the sequence (T_n) where $T_n = 8 \times (\frac{1}{2})^{n-1}$ is

 a) 8 b) 4 c) $\frac{1}{2}$ d) $\frac{1}{8}$

4. If ${}^nC_r = l$ then $l \in$

 a) Z b) Z^+ c) R d) N

Question 2:

1. Some arithmetic means are inserted between 2 and 47. If the ratio between the second and the last means is 2 : 7 what is the number of these means.

2. The third term of a geometric sequence is 9, the sixth term is 243 . Find the sequence, and find the sum of the first eight terms .

Question 3:

1. An arithmetic sequence in which $T_{n+1} - T_n = 2$ if the first term equals 6, find the sequence , then find the sum of the first ten terms.

2. Expand each of the following serieses then find the sum of each expansion.

 a) $\sum_{r=3}^{7} (\frac{1}{3})^{2r+1}$ b) $\sum_{r=1}^{5} (\frac{1}{3})^{r}$

Question 4:

1. In how many ways can a committee of three men and two women formed from four men and three women.

2. Find the value of each of the following

 a) ${}^8P_3$ b) ${}^5P_3 - {}^5P_2$

The fourth test — Algebra

Answer th following questions:

Question 1: Choose the correct answer :-

1. $(T_n) = (3n - 5)$ is an arithmetic sequence of common difference = ... T_2 = ...

 a) 3 , 1 b) 1 , 3 c) 3 , -2 d) 5 , -2

2. The arithmetic mean of two different positive number their geometric mean

 a) > b) < c) = d) ⩽

3. (T_n) is an arithmetic sequence in which $T_n - T_{n+1} = 3$, then the common difference = ...

 a) 3 b) -3 c) 1 d) -1

4. If ${}^nC_3 = 56$ then n = ...

 a) 8 b) 2 c) 10 d) 5

Question 2:

1. Find the number of terms of the arithmetic sequence whose first term is 3 and its last term is 39 and the sum of the first n terms equals 210.

2. In the arithmetic series (3 + 6 + 9 + 12 + ...) find:
 a) The sum of the first 15 terms of the series.
 b) The sum of the terms of the series starting with the fifth term and ending at the fifteenth term.

Question 3:

1. (a , 320 , b , c , 40 , ...) is a geometric sequence of positive terms find the values of a, b, c. then find the sum of infinite number of terms of this sequence starting from the first term.
2. Write three geometric sequences the first term of each of them is 3 such that , one of them is increasing , the second is decreasing and the third is constant , then find the sum of one of them up to infinity.

Question 4:

1. Find the number of two element subset can be formed from the elements of the set {a , b , c , d , e}
2. Find the solution set of each of the following equations:
 a) $\frac{\lfloor n}{\lfloor n-4} = 720$ b) ${}^9C_r = {}^9C_{r-1}$

The fifth test — Algebra

Answer the following questions:

Question 1: choose the correct answer:-

1. The common ratio of the geometric sequence (T_n) where $T_{n-1} = \frac{3}{5}T_n$ is:

 a 3 **b** 5 **c** $\frac{5}{3}$ **d** $\frac{3}{5}$

2. If a , 24 , b are three consecutive terms of an arithmetic sequence , then the value of the expression (a + b) equals:

 a 24 **b** 48 **c** 12 **d** 6

3. The geometric mean of two numbers = 7 then their product = ...

 a 7 **b** 49 **c** 14 **d** –49

4. If ${}^nC_3 = 35$ then n = ...

 a 3 **b** 5 **c** 7 **d** 10

Question 2:

1. In the geometric sequence (2, 4, 8, ...) find :

 a The fifth term **b** the order of the term whose value 512

2. Insert seven geometric means between $\frac{81}{16}$ and $\frac{16}{81}$

Question 3:

1. Find the number of the terms of the arithmetic sequence (8, 11, 14, ..., 50) then find the value of the tenth term from the end.

2. How many terms should be taken from the sequence (27, 24, 21, ...) starting from the first term such that their sum vanishes

Question 4:

1. Find the value of each of the following

 a ${}^6P_3 + {}^6P_1$ **b** $\sum_{r=1}^{5} {}^6C_r$

2. In how many ways can a T-shirt and a pair of pants be dressed from 4 T-shirts and 3 pairs of pants?

The sixth test — Calculus and probability

Answer the following questions:

Question 1: Choose the correct answer

1. The average rate of change of the function $f(x) = x^2$ when x changes from 1 to 2 equals

 a) -3 b) 1 c) 4 d) 3

2. The derivative of the function f where $f(x) = x^2 - 3$ is

 a) $2x$ b) $2x - 3$ c) x^2 d) $3 - 2x$

3. If A, B are mutually exclusive events then $P(A \cap B)$ = ...

 a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) zero d) 1

4. $\int 3x^2 \, dx$ = ...

 a) $x^3 + c$ b) $x^2 + c$ c) $3x^2 + c$ d) x^3

Question 2:

1. If $f(x) = x^2 + 1$ find the rate of change function of f then calculate its value when $x = 1$

2. If A, B are two mutually exclusive events, $P(A) = 0{,}4$, $P(B) = 0{,}3$ find $P(A \cup B)$

Question 3:

1. If $y = (x^2 + 1)(x^2 + 3)$ find $\frac{dy}{dx}$

2. In the experiment of rolling a single die and observing the number on the upper face. Write the sample space then identify the following events showing the number of elements of each.

 - the event A «appearance of a prime number »
 - the event B «appearance of a number divisible by 6 ».
 - The event C « appearance of a number divisible by 7»

Question 4

1. Find the measure of the angle which the tangent to the curve $y = x^2 - 2x$ at the point (1, -1) makes with the positive direction of x-axis.

2. Find:

 a) $\int (2x + 3) \, dx$ b) $\int \frac{x^3 + 1}{x + 1} \, dx$

The seventh test — Calculus and probability

Answer the following questions:

Question 1: Choose the correct answer :

1. If $f(x) = 2x$ then $\int f(x)\, dx$

 a) x^2 b) $x^2 + c$ c) $2x$ d) $2x + c$

2. The derivative of the function f where $f(x) = \frac{1}{x}$ equals

 a) $\frac{1}{x^2}$ b) $\frac{-1}{x^2}$ c) x d) x^2

3. In the experiment of rolling a die twice, the probability of getting two equal numbers is:

 a) $\frac{1}{5}$ b) $\frac{1}{36}$ c) $\frac{1}{6}$ d) $\frac{1}{18}$

4. The measure of the positive angle which the tangent to the curve $y = x^2 - 1$ at $x = \frac{1}{2}$ makes with the positive direction of x-axis

 a) $\frac{\pi}{4}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{6}$ d) π

Question 2:

1. Find the rate of change of the function f where $f(x) = 2x - x^2$ when $x = 2$

2. A box contain 6 identical cards numbered from 1 to 6 one card is selected at random and the number on the card is observed. Find the probability of each of the following events:

 - A is the event of getting a prime number .
 - B is the event of getting an even number divisible by 3.

Question 3

1. Find the first derivative of the function $y = (x - 1)(x + 1)$

2. If A, B are two events of a sample space of a random experiment , :
 $P(A) = \frac{1}{3}$, $P(B) = \frac{3}{8}$, $P(A \cap B) = \frac{1}{4}$ Find

 a) $P(A')$ b) $P(A \cup B)$

Question 4

1. Find the points on the curve $y = x^2 + 2$ at which the slope of the tangent equals 2

2. Find each of the following:

 a) $\int -2z\ dz$ b) $\int ((x - 1)^5 + 3)\, dx$

The eighth test — Calculus and probability

Answer the following questions:

Question 1: Choose the correct answer:

1. The rate of change of the function f where: $f(x) = x^2 - 1$ at $x = -2$ equals
 a) 4 b) -4 c) -1 d) 1

2. $\int x^{-2}\, dx$ = ...
 a) $-x^{-1} + c$ b) $-\frac{1}{x}$ c) $1 + \frac{1}{x^2}$ d) $x^{-3} + c$

3. The two events A and B are said to be mutually exclusive if $P(A \cap B)$ = ...
 a) zero b) 1 c) $\frac{1}{2}$ d) P(A)

4. The slope of the tangent to the curve $y = \frac{1}{x}$ when $x = 1$ equals
 a) 1 b) -1 c) zero d) undefined

Question 2:

1. If $f(x) = 2x - x^2$ find :
 a) The variation function V(h) when $x = 1$
 b) The average rate of change when x changes from 1 to 1.1

2. A , B are two events of a sample spaces of a random experiment , P is a probability function defined on S where:
 $P(B) = 3P(A)$, $P(A \cup B) = 0.72$ find P(A), P(B) If:
 a) A and B are mutually exclusive events b) $A \subset B$

Question 3:

1. If $y = 4x^3 - 5x^2 + 4x + 9$ find $\frac{dy}{dx}$

2. A bag contains 4 white balls, 5 red balls and 3 black balls one ball is selected at random. Find the probability that the selected ball is:
 a) Red b) not white c) white or black

Question 4:

1. Find the points on the curve $y = x^3 + 3x^2 - 1$ at which the tangent is parallel to x-axis.

2. Find the following integrals:
 a) $\int (x - 5)(x - 1)\, dx$ b) $\int 6(x - 2)^5\, dx$

The ninth test — Calculus and probability

Answer the following question:

Question 1: choose the correct answer:

1. The rate of change of the function $f(x) = x^2$ at $x = 1$ equals:

 a) 2 b) 1 c) -1 d) -2

2. The probability of the certain event equals :

 a) zero b) 1 c) $\frac{1}{2}$ d) -1

3. The slope of the tangent to the curve $y = \sqrt[3]{x^2}$ at $x = 1$ equals

 a) zero b) $\frac{2}{3}$ c) $\frac{3}{4}$ d) 1

4. $\int (x^2 - 1)\, dx$

 a) $x^3 - x$ b) $\frac{1}{3}x^3 - x + c$ c) $\frac{1}{3}x^3 + c$ d) $- x + c$

Question 2:

1. Find the rate of change of the function f where: $f(x) = x + \frac{1}{x}$ at $x = 2$

2. If $S = \{a, b, c\}$ is the sample space of a random experiment P is the probability function on this experiment Find : P(A) If $P(B) = 2P(A)$, $P(C) = \frac{1}{4}$ then find $P(A \cup C)$

Question 3:

1. Find the first derivative of each of the following:

 a) $y = \frac{x}{x+1}$ b) $y = x^3 + x^2 - 1$

2. If A is the event that a student success in physics, B is the event that the student success in math and if $P(A) = \frac{1}{2}$, $P(B) = \frac{2}{3}$, $P(A \cap B) = \frac{1}{3}$

 a) Calculate the probability that the student success in at least one of the two subjects.

 b) Calculate the probability that the student success in physics only.

Question 4:

1. Find the equation of the tangent to the curve of the function $y = (x - 1)(x + 1)$ at the points of intersection of the curve with x-axis.

2. Find:

 a) $\int (2x - 3)^5\, dx$ b) $\int 2x(x^2 + 3)\, dx$

The tenth test — Calculus and probability

Answer the following question:

Question 1: choose the correct answer:

1. $\frac{d}{dx}(\frac{1}{x^2}) =$

 a x^{-2} b $\frac{-2}{x^3}$ c $-2x^{-2}$ d x^{-3}

2. The average rate of change of the function f where $f(x) = \sqrt{x}$ when x changes from 4 to 4.41 equals

 a 40 b 0.41 c $\frac{41}{10}$ d $\frac{10}{41}$

3. $\int 5\, dx =$

 a 5 b $5x + c$ c $5x$ d $5 + x$

4. The slope of the tangent to the curve $y = 2x^2$ at the point (1, 2) equals

 a 2 b 1 c 4 d zero

Question 2:

1. If $y = (3x - 1)^5$. Find $\frac{dy}{dx}$

2. A regular coin is tossed three times in succession and the sequence of heads and tails are observed calculate the probability of each of the following events:

first: A is the event of getting only one head. second: B is the event of getting at least two heads.

Question 3:

1. Find the first derivative of each of the following function:-

 a $f(x) = \frac{1}{x}$ b $f(x) = x^2 + x$

2. A secondary school has 20 teacher and 4 administrators, two persons are randomly choosen to escort the student in a mission . what is the probability that the two escorts are:

 a Teachers b Administrator and a teacher c Ademonstrators.

Question 4:

1. Find the slope of the tangent to the curve $y = \frac{x^2 + 1}{x}$ at the point (1, 2)

2. Find :

 a $\int \frac{x^2 - 5x + 6}{x - 2}\, dx$ b $\int 4(x - 2)^3\, dx$

Answer of some exercises

Unit one: counting principle permutations and combinations

Answer of some exercise (1-1)

1) 9 2) 19 3) 1 4) 19 5) a 6) d

7) a infinite b finite c infinite d finite

8) a (2, 6, 12, 20, 30)
b $(-\frac{1}{3}, -1, 1, \frac{1}{3}, \frac{1}{5})$
c $(\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243})$
d (-1, 0, -1, 4, -9)

9) a Pattern $(T_n) = 4n + 61$
b $(T_n) = 3(-2)^{n-1}$
c $(T_n) = (\frac{1}{2})^n$

10) a (8 , 10 , 12 , 14 , 16)
b $T_N = 2n + 6$ c 20
d The twenty day

11) a (1 + 4 + 7 + 10 + 13)
b (3 + 9 + 11 + 17 + ... to 8 terms)
c $(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \infty)$
d $(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots \infty)$

12) a 65 b (2 + 7 + 14 + 23 + 34) 80

14) a $(3 \times 32 + 3 \times 16 + 3 \times 8 + 3 \times 4)$
b The perimeter of the fourth triangle = 12cm
c 180 cm

15) a $(3^1 + 3^2 + 3^3 + 3^4 + 3^5 + \ldots) = \sum_{1=r}^{n} 3^n$
b 243 c 363

Answer of some exercises (1 - 2)

1) Arithmetic sequence d = 3
2) Arithmetic sequence
3) Arithmetic sequence d = 6
4) Constant arithmetic sequence d = 0
5) Arithmetic sequence $d = 2x + y$
6) (2 , 7 , 12 , ...) 7) (7 , 4 , 1 , ...)
8) $(-4, -\frac{15}{4}, -\frac{7}{2}, \ldots.)$
9) 20 10) 28
11) $T_n = 85 - 84$ 12) $T_n = \frac{3}{4} - \frac{1}{4}n$
13) 10 14) 15
15) c 16) d
17) d 18) 3

Answer of some exercise (1-3)

1) 210 2) 110 3) 200
4) 243 5) 90 6) 35
7) 24 8) $\sum_{r=1}^{n} (52 + r5)$ $\sum_{r=1}^{\xi}$ 9) (1 - r)
10) False, the correct is the sum = 0
11) False, the correct is the sum of the first nine terms = 0 or sum of the first six terms = 204
12) $a = 14, d = 4, S_n = \frac{10}{2}[2 \times 14 + 9 \times 4] = 320$
13) $S_{15} = \frac{15}{2}(a + l) = \frac{15}{2}(4 + 26) = 225$
14) (420)
15) - 260 16) 1020 17) 1107

Answer of some exercise (1-4):

1) Not geometric sequence
2) Geometric sequence $r = \frac{1}{3}$
3) Geometric sequence 2 = r
4) Geometric sequence $r = \frac{1}{2}$
5) Geometric sequence -3 = r
6) not geometric sequence

Complet:

11) one 12) one 13) 162
14) $T_n = 3(-2)^{n-1}$ 15) ±8 16) 25

Choose:

18) d 19) b 20) b 21)

Answer of some exercises (1 - 5):

1) b 2) a 3) c 4) c 5) a

Answer some of general exercises

1) 1 2) $\sum_{1=r}^{10} 5r$ 3) $T_n = 4n - 1$

(4) ± 12 (5) a = 7 (6) 19
(7) 80 (8) 31

Answer of accumulative test

(1) The sequence is the order of a set of the real number.

(2) The n^{th} term of the arithmetic sequence in the form of $\mathbf{T}_n = a + (n - 1)\,\mathbf{d}$ where a the first term , d the common difference.
The n^{th} term of the geometric sequence $T_n = ar^{-1}$ n where a the first term and r is the common ratio.

Short- answered questions

(4) **a** 14 **b** $\frac{1}{2}$
c -15 **d** 1

Unit 2: Counting principle, permutations and combinations

Answers of exercises (2 - 1):

(1) d (2) c (3) d
(4) $3^3 = 27$ (5) 6
(8) Number of digits = $(10)^8 = 100000000$

Answers of exercises (2 - 2):

(1) d (2) b (3) a
(4) d (5) c (6) c
(7) b (8) 6 (9) 16
(10) $^4l_2 = 12$
(12) Number of ways = $3 \times 4 = 12$
(13) Number of ways of choosing = 15
(14) **a** $^6l_1 = 6$ **b** $^6l_2 = 30$
(15) **a** 42 **b** 0
c 120 **d** 12
e 64 **f** 5041
(16) **a** 4 **b** 6
c n = 3 **d** 7
(17) **a** $^7l_n = {}^7l_3$, n = 3
b $\lfloor 2n = 24$, n = 2
(18) n = 7 or n = 8
(19) $^{10}l_3 = 720$
(20) $^8l_3 = 336$
(21) Right = (n + 2) (n + 1) = $^{n+2}l_2$

Answers of exercises (2 - 3):

(1) b (2) b (3) a
(4) 20 , 9 , 12 , 1
(5) 10 (6) 9 (7) 15
(8) $^5C_5 + {}^5C_4 + {}^5C_3 = 16$
(9) $^{10}C_3 \times {}^8C_2 = 3360$
(10) **a** $\dfrac{^8l_3}{\lfloor 3}$ **b** $\dfrac{^{19}l_2}{\lfloor 2}$
c $\dfrac{^nl_1}{\lfloor 0}$ **d** $\dfrac{^xl_y}{\lfloor y}$
(11) **a** 28 **b** 84
c 210 **d** 1

Answers of general exercises

(1) $^4l_2 = 4 \times 3 = 12$

(2) $^39 = 729$

(3) $^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^n$
$^42 = 2^n$ $\therefore$ n = 4

(4) $\because \lfloor n = \lfloor 1$, $\therefore$ n = 0 or n = 1

(5) **b** (6) **c**

(7) **c** (8) **a**

(9) **b** (10) **b**

(11) **a** 1 **b** 720 **c** 23
d 50 **e** $\frac{1}{14}$ **f** 24

(12) n = 5 , $\lfloor n-5 = 1$ (13) r = 2 , $^{2r}C_4 = 1$
(14) m = 3 , n =10 (15) $^{10}C_7 = 120$
(16) n = 5
(17) $^7C_1 \times {}^5C_2 = 7 \times 10 = 70$

(18) 12

(19) $^{12}P_2 = 132$

20 a n = 4 b n = 2 c n = 9
d n = 5 or n = 6 e n = 9 f r= 3

Answer of accumulative test

1 d 2 c 3 c 4 b
5 b
6 a 32 b 0
7 400 8 10
9 120
10 a $\frac{1}{100}$ b 10 c $\frac{1}{20}$
d 96 e 24
11 ${}^6C_2 = 15$
12 a n = 6 b n = $\frac{5}{2}$
c n = 8 d n = 6
e n = 1 f n = 4

Unit 3 : Calculus

Answers of exercises (3 - 1):

1 b 2 d 3 d 4 c

5 $d(x) = x^2 + 2x - 1$
$\because$ when x varies from x_1 to $x_1 + h$
$\therefore F(h) = f(x_1 + h) - f(x_1)$
$= (x_1 + h)^2 + 2(x_1 + h) - 1 - x_1^2 - 2x_1 + 1$
$= 2x_1 h + h^2 + 2h$

a 0,61
b 1

Answers of exercises (3 - 2)

1 Using the definition of derivative function, we find that:
a $f'(x) = 5$ b $f'(x) = 6x$
c $f'(x) = 3x^2$ d $f'(x) = 2x + 2$

2 a $f'(x) = \frac{-1}{x^2}$

The function f is non differentiable when x = 0

b $d'(x) = \frac{-1}{(x+3)^2}$

f is non differentiable when x = -3

c $d'(x) = \frac{-6}{(2x-5)^2}$

f is non differentiable when $x = \frac{5}{2}$

d $d'(x) = \frac{1}{2\sqrt{x-4}}$

f is non differentiable when $x \geq 4$

Answers of exercises (3 - 3):

1 d 2 a 3 c 4 d
5 2 6 6x 7 $4x^3 - 4x$
8 $1 + \frac{1}{2\sqrt{x}}$ 9 $\frac{-3}{x^4}$ 10 zero
11 $\frac{1}{2}x^{-\frac{1}{2}}$ 12 $\frac{-1}{3\sqrt[5]{x^4}}$ 13 10x + 3
14 $7\sqrt{2}x^6 - x^4$
15 a $15x^4$ b $-6x^{-9}$ c $\frac{-3}{x^3}$
d $\frac{2}{\sqrt{x}}$ e $\frac{5}{3}\sqrt[5]{x^2}$
f $\frac{-1}{x\sqrt{x}}$
ز $\frac{3}{2}\sqrt{x}$ ح $\frac{3}{2}\sqrt{x}$
16 $3x^2 + 6x$ 17 $2x^3 - 2x^2 + 7$
18 $12x^5 + \frac{3}{2\sqrt{x}}$ 19 $4 - \frac{1}{x\sqrt{x}}$
20 $9x^2 - \frac{3}{2}\sqrt{x}$
21 $6\sqrt{x} - \frac{1}{2\sqrt{x}} - \frac{3}{2\sqrt{x^3}}$
22 $6x^2 + 10x + 2$ 23 $6x^2 + 6x - 14$
24 $5x^4 + 2x - 9$ 25 $4x^3 + \frac{5}{2}x2 - \frac{3}{2}$
26 $\frac{15}{(5x+1)^2}$ where $x \neq -\frac{1}{5}$
33 $y - 5x + 10 = 0$, $5y + x - 2 = 0$
35 C = 3 , b = -4

Answers of exercises (3 - 4)

1 $\frac{1}{3}x^3 + C$ 2 $\frac{1}{8}x^8 + C$
3 $4x^2 + C$ 4 $-x^4 + C$

(5) $x^9 + C$ (6) $-\frac{4}{x^3} + C$

(7) $5x + C$ (9) $-\frac{3}{8}x^8 + C$

(10) $\frac{4}{5}x^{-2} + C$ (11) $\frac{1}{3}\ n^7 + C$

(12) $\frac{2}{5}\ g^6 + C$ (13) $\frac{1}{2}x^2 + x + C$

(14) $5n - n^2 + C$ (15) $\frac{1}{4}\ n^4 - \frac{6}{5}\ n^5 + C$

(20) $x^7 - \frac{7}{3}x^3 + C$ (21) $\frac{1}{3}x^3 - x + C$

(27) $\frac{1}{2}x^2 + x + C$ (28) $\frac{1}{3}x^3 + \frac{1}{2}x^2 + 9x + C$

(29) $\frac{1}{2}x^2 + 2x + C$ (30) $\frac{1}{2}x^2 - 2x + C$

(31) $\frac{1}{4}(x+4)^4 + C$ (32) $\frac{1}{2}(2x-7)^7 + C$

(33) $\frac{-1}{15}(8 - 3n)^5 + C$

(34) $-2(x-3)^{-3} + C$

Answer some of general exercises :

(10) (a) $\frac{dy}{dx} = 2x - 3$

(b) $\frac{dy}{dx} = 15x^2 + 6x - 4$

(c) $\frac{dy}{dx} = 8x^3 - 5x^2 + 7$

(d) $\frac{dy}{dx} = 6x^2 + \frac{3}{x^2} + \frac{2}{x^3}$

(11) (a) 0 (b) 1 (c) 4 (d) 2

(12) (a) 45° (b) 45° (c) 135° (d) 20′ 96°

(14) 2 (15) 0

(16) (a) $\frac{dy}{dx} = 6(2x+1)^2$

(b) $\frac{dy}{dx} = 15x^2(x^3+1)^4$

(c) $\frac{dy}{dx} = 4(6x - 1)(3x^2 - x + 2)^3$

(d) $\frac{dy}{dx} = 24x^2(2x^3+5)^3$

(17) $\frac{1}{3}x^3 - \frac{3}{2}x^2 - x + C$

(18) $\frac{1}{3}\sqrt{x^3} + 4\sqrt{x} + C$

(19) $\frac{1}{3}x^3 - 4x + C$

(20) $\frac{1}{3}x^3 - 2x^2 - 5x + C$

(21) $\frac{1}{2}x^2 + x + C$

(22) $\frac{1}{3}x^3 + \frac{3}{2}x^2 + 9x + C$

(23) $(x-2)^6 + C$

(24) $-(x+3)^{-3} + C$

Answer of accumulative test

(1) b (2) c (3) d (4) c

(5) (a) $6x - 5$ (b) $3x^2 - 6x + 5$

(c) $2x$ (d) $\frac{2}{(x+1)^2}$

Unit 4: Probability

Answers of exercises (4 - 1)

(5) A = { 12, 14, 16, 18, 20} , B = { 1, 2, 3, 4, 6 , 12}

C = { 3, 9, 15} , D = { 10 ,20} , F = { 1 , 2, 3, 4}

(9) c (10) d (11) b (12) d (13) d

(15) $P(a) = \frac{1}{6}$, $P(b) = \frac{1}{18}$

(18) (a) 0,2 (b) 0,45 (c) 0,55

(22) $\frac{1}{6}, \frac{5}{36}, \frac{5}{18}$ (23) $\frac{13}{15}, \frac{4}{15}$, 0,4 , 0,6

Answer of some exercises

(1) c (2) d (3) c (4) a (5) a

(6) d (7) c (11) l (b́) = 0,3

(14) $P(a) = \frac{3}{8}$ $P(b) = \frac{7}{8}$ $P(c) = \frac{1}{2}$ $P(d) = \frac{1}{4}$

(17) (a) 0,36 (b) 0,84

(18) (a) 0,235 (b) 0,1 (c) 0,53 (d) 0,825

Accumulative test

(1) g = {1, 2, 3, 4, 5, 6}

(2) { (y, k, (k,y), (k, k)}

(3) {(2,y), (2, k), (3,y) , (3, k), (5,y), (g, k)}

(4) { (1, 4), (2, 3), (3, 2), (4,1)}

(5) { 3, 6, 9, 12, 15, 18}

(6) { (y,y, k), (y, k,y), (k,y,y)}

(8) (a) 0,15 (b) 0,2 (c) 0,3

(9) (a) 0,3 (b) 0,45 (c) 0,7

10 a 0,25 b 0,35 c 91

Test 1 :

Question 1:

1 b 2 c 3 d 4 c

Question 2:

1 n = 7 2 2

Question 3:

1 a (16, 32, 64, ...) b 16368

2 i (-3) + (-5) + (-7) + (-9) +(-11) +(-13) = -48

Question 4:

1 a 114 b 240

2 a r = 3 b 8 or 9

Test 2 :

Question 1:

1 c 2 c 3 a 4 b

Question 2:

1 (5, 8, 11, 14, 17, 20, 23, 26) , the sum = 124

2 48

Question 3:

1 762 2 a 1,4999 b 133

Question 4:

1 a 118 b 14

2 $\lfloor 5$ = 120

Test 3:

Question 1:

1 a 2 c 3 d 4 b

Question 2:

1 8 2 (1, 3, 9, ...) , 3280

Question 3:

1 (6, 8 , 10, ...) , 150

Question 4:

1 ${}^4C_3 \times {}^3C_2 = 12$ 2 a 336 b 40

Test 4:

Question 1:

1 a 2 a 3 b 4 a

Question 2:

1 10 2 a 360 b 330

Question 3:

1 a = 640 b = 160

S = 80 the sum to infinity = 1280

2 (3, 5, 7, ...) , $(3, \frac{3}{2}, \frac{3}{4}, ...)$, (3, 3, 3, ...)

$S_\infty = 6$

Question 4:

1 ${}^5C_2 = 10$ 2 a n = 10 , b r = 5

Test 5:

Question 1:

1 c 2 b 3 b 4 c

Question 2:

1 a 32 b 9 2 $(\frac{27}{8}, \frac{3}{4}, ..., \frac{8}{27})$

Question 3:

1 n = 15 , T_{10} from the end = 23 2 n = 19

Question 4:

1 a 126 b 62C 2 12 ways

Test 6

Question 1:

1 d 2 a 3 c 4 a

Question 2:

1 2 2 0,7

Question 3:

1 $4x^3 + 8x$

Question 4:

(1) zero (2) (a) $x^2 + 3x + C$

(b) $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + C$

Test 7:

Question 1:

(1) b (2) b (3) c (4) a

Question 2:

(1) -2 (2) $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{6}$,

Question 3:

(1) $\frac{dy}{dx} = 2x$ (2) (a) $\frac{2}{3}$ (b) $\frac{11}{24}$

Question 4:

(1) (1, 3) (2) (a) $-Z^2 + C$

(b) $\frac{1}{6}(x-1)^6 + 3x + C$

Test 8 :

Question 1:

(1) b (2) a (3) a (4) b

Question 2:

(1) (a) $-h^2$ (b) -0,1

Question 3:

(1) $y = 12x^2 - 10x + 4$ (2) $\frac{5}{12}, \frac{2}{3}, \frac{7}{12}$

Question 4 :

(1) (0, -1) , (-2, 3)

(2) (a) $x^3 - 3x^2 + 5x + C$ (b) $(x-2)^6 + C$

Test 9:

Question 1:

(1) a (2) b (3) b (4) b

Question 2:

(1) $\frac{3}{4}$ (2) $\frac{1}{4}, \frac{1}{2}$

Question 3:

(1) (a) $\frac{-1}{(x+1)^2}$ (b) $3x^2 + 2x$

(2) (a) $\frac{5}{6}$ (b) $\frac{1}{6}$

Question 4:

(1) $y = 2x - 2$, $y = -2x - 2$

(2) (a) $\frac{1}{12}(2x-3)^6 + C$ (b) $\frac{1}{2}x^4 + 3x^2 + C$

Test 10:

Question 1:

(1) c (2) d (3) b (4) c

Question 2:

(1) $15(3x-1)^4$ (2) $P(a) = \frac{3}{8}$ $P(b) = \frac{1}{2}$

Question 3:

(1) (a) $\frac{-1}{x^2}$ (b) $2x + 1$

(2) (a) $\frac{95}{138}$ (b) $\frac{20}{69}$ (c) $\frac{1}{46}$

Question 4:

(1) zero (2) (a) $\frac{1}{2}x^2 - 3x + C$ (b) $(x-2)^4 + C$